

Rate of Convergence of the Expected Spectral Distribution Function to the Marchenko – Pastur Law

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Abstract

Let $\mathbf{X} = (X_{jk})$ denote a $n \times p$ random matrix with entries X_{jk} , which are independent for $1 \leq j \leq n, 1 \leq k \leq p$. Let n, p tend to infinity such that $\frac{n}{p} = y + O(n^{-1}) \in (0, 1]$. For those values of n, p we investigate the rate of convergence of the expected spectral distribution function of the matrix $\mathbf{W} = \frac{1}{p}\mathbf{X}\mathbf{X}^*$ to the Marchenko-Pastur law with parameter y . Assuming the conditions $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$ and

$$\sup_{n,p \geq 1} \sup_{1 \leq j \leq n, 1 \leq k \leq p} \mathbf{E}|X_{jk}|^4 =: \mu_4 < \infty, \quad \sup_{n,p \geq 1} \sup_{1 \leq j \leq n, 1 \leq k \leq p} |X_{jk}| \leq Dn^{\frac{1}{4}},$$

we show that the Kolmogorov distance between the *expected* spectral distribution of the sample covariance matrix \mathbf{W} and the Marchenko – Pastur law is of order $O(n^{-1})$.

1 Introduction

The present paper is a continuation of the papers [15], [16], where we proved non improvable bounds for the Kolmogorov distance between the expected

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spectral distribution function of Wigner matrices and the semicircular distribution function. In this paper we estimate the Kolmogorov distance between the expected spectral distribution function of sample covariance matrices and the Marchenko – Pastur distribution function.

Consider a family $\mathcal{X} = \{X_{jk}\}$, $1 \leq j \leq n, 1 \leq k \leq p$, of independent real random variables defined on some probability space $(\Omega, \mathfrak{M}, \Pr)$, for any $n \geq 1$ and $p \geq 1$. Introduce the matrices

$$\mathbf{X} = \frac{1}{\sqrt{p}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{pmatrix}$$

and corresponding sample covariance matrices

$$\mathbf{W} = \mathbf{X}\mathbf{X}^*.$$

Here and in what follows we denote by \mathbf{A}^* the complex conjugate of matrix \mathbf{A} . The matrix \mathbf{W} has a random spectrum $\{s_1^2, \dots, s_n^2\}$ and an associated spectral empirical distribution function $\mathcal{F}_n(x) = \frac{1}{n} \text{card} \{j \leq n : s_j^2 \leq x\}$, $x \in \mathbb{R}$. Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions $F_n(x) = \mathbf{E} \mathcal{F}_n(x)$. Let $G_y(x)$ denote the Marchenko – Pastur distribution function with parameter y with density $g_y(x) = G'_y(x) = \frac{1}{2\pi x} \sqrt{(x - a^2)(b^2 - x)} \mathbb{I}_{[a^2, b^2]}(x)$, where $\mathbb{I}_{[a^2, b^2]}(x)$ denotes the indicator-function of the interval $[a^2, b^2]$ and $a^2 = (1 - \sqrt{y})^2$ and $b^2 = (1 + \sqrt{y})^2$. The rate of convergence to the Marchenko – Pastur law has been studied by several authors. For a detailed discussion of previous results see [15] and [13]. In what follows we shall assume that $p = p(n)$ such that

$$y_n := \frac{n}{p}, \quad |y_n - y| \leq c_y n^{-1}, \quad (1.1)$$

for some constant $c_y > 0$. We shall estimate the Kolmogorov distance between $\mathcal{F}_n(x)$ and the distribution function $G_y(x)$, that is, $\Delta_n := \sup_x |F_n(x) - G_y(x)|$.

The main result of this paper is the following

Theorem 1.1. *Let $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. Assume that there exists a constant $\mu_4 > 0$ such that*

$$\sup_{n, p \geq 1} \sup_{1 \leq j \leq n, 1 \leq k \leq p} \mathbf{E}|X_{jk}|^4 =: \mu_4 < \infty. \quad (1.2)$$

Furthermore, assume that there exists a constant D such that for all n

$$\sup_{1 \leq j \leq n, 1 \leq k \leq p} |X_{jk}| \leq Dn^{\frac{1}{4}}. \quad (1.3)$$

Assuming (1.1), for $y \in (0, 1]$ there exists a positive constant $C = C(D, \mu_4, y, c_y)$ depending on D , μ_4 , y and c_y only such that,

$$\Delta_n = \sup_x |F_n(x) - G_y(x)| \leq Cn^{-1}. \quad (1.4)$$

Corollary 1.1. Let $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. Assume that

$$\sup_{n \geq 1} \sup_{1 \leq j \leq k \leq n} \mathbf{E}|X_{jk}|^8 =: \mu_8 < \infty. \quad (1.5)$$

Assuming (1.1), for any $y \in (0, 1]$ there exists positive constants $C = C(\mu_8, y, c_y)$ depending on μ_8 , y and c_y only such that,

$$\Delta_n \leq Cn^{-1}. \quad (1.6)$$

Remark 1.2. It is straightforward to check that by assumption (1.1)

$$\sup_x |G_{y_n}(x) - G_y(x)| \leq Cn^{-1}, \quad (1.7)$$

with a constant C depending on y and c_y . Thus without loss of generality we shall assume in the following proofs that $y = y_n$.

For any distribution function $F(x)$ we define the Stieltjes transform $s_F(z)$, for $z = u + iv$ with $v > 0$, via formula

$$s_F(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x). \quad (1.8)$$

We introduce the symmetrized distribution function

$$\tilde{\mathcal{F}}_n(x) = \frac{1 + \text{sign}(x)\mathcal{F}_n(x^2)}{2}.$$

We denote the Stieltjes transform of $\mathcal{F}_n(x)$ by $m_n(z)$ and the Stieltjes transform of the Marchenko – Pastur law with parameter y by $S_y(z)$. Let $\mathbf{R} = \mathbf{R}(z)$ be the resolvent matrix of \mathbf{W} given by $\mathbf{R} = (\mathbf{W} - z\mathbf{I}_n)^{-1}$, for all $z = u + iv$ with $v \neq 0$. Here and in what follows \mathbf{I}_n denotes the identity

matrix of dimension n . Sometimes we shall omit the sub-index in the notation of an identity matrix. Denote by $m_n(z)$ the Stieltjes transform of the distribution function $\mathcal{F}_n(x)$. It is a well-known fact that

$$m_n(z) = \frac{1}{n} \sum_{j=1}^n \frac{1}{s_j^2 - z} = \frac{1}{n} \text{Tr } \mathbf{R}. \quad (1.9)$$

The Stieltjes transform $S_y(z)$ of the Marchenko – Pastur distribution satisfies the equation

$$yzS_y^2(z) + (y - 1 + z)S_y(z) + 1 = 0 \quad (1.10)$$

(see, for example, equality (3.10) in [12]). For the Stieltjes transform $s_y(z) = zS_y(z^2)$ of the symmetrized Marchenko – Pastur distribution we have

$$1 + (z + \frac{y-1}{z})s_y(z) + ys_y^2(z) = 1 \quad (1.11)$$

(see, for instance, equality (3.11) in [12]). We introduce the $(n+p) \times (n+p)$ Hermitian matrix by

$$\mathbf{V} = \begin{bmatrix} \mathbf{O} & \mathbf{X} \\ \mathbf{X}^* & \mathbf{O} \end{bmatrix}.$$

It is well known that the eigenvalues of the matrix \mathbf{V} are $-s_1, \dots, -s_n, s_n, \dots, s_1$ and 0 of multiplicity $p - n$. Introduce the resolvent matrix $\tilde{\mathbf{R}}$ of the matrix \mathbf{V} by

$$\tilde{\mathbf{R}} = (\mathbf{V} - z\mathbf{I}_{n+p})^{-1}. \quad (1.12)$$

It is straightforward to check that

$$\tilde{m}_n(z) = zm_n(z^2) = \frac{1}{2n} \text{Tr } \tilde{\mathbf{R}} + \frac{1-y}{2zy}. \quad (1.13)$$

In what follows we shall consider the symmetrized distribution only. If it is clear from the context we shall omit the symbol \sim in the notation of distribution functions, Stieltjes transforms, resolvent matrices and etc.

Let

$$v_0 := A_0 n^{-1} \quad (1.14)$$

and $\gamma(z) := \min\{|a - |u||, |b - |u||\}$, for $z = u + iv$. Introduce the region $\mathbb{G} = \mathbb{G}(A_0, n, \varepsilon) \subset \mathbb{C}_+$

$$\mathbb{G} := \{z = u + iv \in \mathbb{C}_+ : a + \varepsilon \leq |u| \leq b - \varepsilon, V \geq v \geq v_0 / \sqrt{\gamma(z)}\}.$$

Let $\varkappa > 0$ be a positive number such that

$$\frac{1}{\pi} \int_{|u| \leq \varkappa} \frac{1}{u^2 + 1} du = \frac{3}{4}. \quad (1.15)$$

On the level of Stieltjes transforms our results are based on the following approximations.

Theorem 1.3. *Let $\frac{1}{2} > \varepsilon > 0$ be positive numbers (depending on v_0 , see (1.14)) such that*

$$\varepsilon^{\frac{3}{2}} := 2v_0\kappa. \quad (1.16)$$

Assuming the conditions of Theorem 1.1, there exists a positive constant $C = C(D, A_0, \mu_4, y)$ depending on D , A_0 , μ_4 and y only, such that, for $z \in \mathbb{G}$

$$|\mathbf{E}m_n(z) - s_y(z)| \leq \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}.$$

For the readers convenience we decided to exposed the full proof for the Marchenko-Pastur case again since in nearly any of the arguments of the 60 page proof of the Wigner result in [15] adjustments and rewritings for this case had to be done. For more details see the sketch of the proof below.

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2 Sketch of the Proof

1. As in previous work [12] we use the so called Hermitization of a matrix \mathbf{X} and instead of the spectrum of the matrix \mathbf{W} we consider the spectrum of the block-matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{O} & \mathbf{X} \\ \mathbf{X}^* & \mathbf{O} \end{bmatrix},$$

where \mathbf{O} denotes the matrix with zero entries. Thus the proof will be very similar to the proof for the expected convergence to the Wigner law in [15] but using resolvent identities reflecting the jump of size $1 - y$ at zero in the spectrum. Notable larger changes appear in Section 7 (see point 4 below).

2. Furthermore, similarly to [15], we start with an estimate of the Kolmogorov-distance to the symmetrized Marchenko – Pastur distribution via an integral over the difference of the corresponding Stieltjes transforms along a contour in the upper half-plane using a smoothing inequality (3.6). The resulting bound (3.6) involves an integral over a segment, say $V = 4\sqrt{y}$, at a fixed distance from the real axis and a segment $u + iA_0n^{-1}(\min\{(a - |u|), (b - |u|)\})^{-\frac{1}{2}}$, $u + iV$, $a + \varepsilon \leq |u| \leq b - \varepsilon$ at a distance of order n^{-1} but avoiding to come close to the endpoints a and b of the support. These segments are part of the boundary of an n -dependent region \mathbb{G} where bounds of Stieltjes transforms are needed. Since the Stieltjes-transform and the diagonal elements $R_{jj}(z)$ of the resolvent $(\mathbf{R} = (\mathbf{V} - z\mathbf{I}_{n+p})^{-1})$ of the matrix \mathbf{V} are uniformly bounded on the segment with $\text{Im } z = V$ by $1/V$ (see Section 4.1) proving a bound of order $O(n^{-1})$ for the latter segment near the x -axis is the essential problem.

3. In order to investigate this crucial part of the error we start with the 2nd resolvent or self-consistency equation for the Stieltjes transform resp. the quantities $R_{jj}(z)$ of \mathbf{V} (see (6.32) below) based on the difference of the resolvent of $\mathbf{V}^{(j)}$ (j th row and column removed) and \mathbf{V} . The necessary bounds of $\mathbf{E}|R_{jj}|^q$ for large $q = O(1)$ we prove analogously to [15].

4. In Section 7 we prove a bound for the error $\mathbf{E}\Lambda_n = \mathbf{E}m_n(z) - s_y(z)$ of the form $n^{-1}v^{-\frac{3}{4}} + (nv)^{-\frac{3}{2}}|(z + \frac{y-1}{z})^2 - 4y|^{-\frac{1}{4}}$ which suffices to prove the rate $O(n^{-1})$ in Theorem 1.1. Here we use a series of martingale-type decompositions to evaluate the *expectation* $\mathbf{E}m_n(z)$ combined with the bound $\mathbf{E}|\Lambda_n|^2 \leq C(nv)^{-2}$ of Lemma 8.17 in the Appendix which is again based on a recursive inequality for $\mathbf{E}|\Lambda_n|^2$ in (8.57). A direct application of this bound to estimate the error terms ε_{j3} would result in a less precise bound of order $O(n^{-1} \log n)$ in Theorem 1.1. Bounds of such type will be shown for the Kolmogorov distance of the empirical *random* spectral distribution

to Marchenko – Pastur law in a separate paper. For the expectation we provide sharper bounds in Section 7.2 involving $m'_n(z)$. Note here that in the Marchenko–Pastur case a new term c/z^2 appears.

5. The necessary auxiliary bounds for all these steps are collected in the Appendix.

3 Bounds for the Kolmogorov Distance Between Distribution Functions via Stieltjes Transforms

To bound Δ_n we shall use an approach developed in Götze and Tikhomirov [12] and [10]. We modify a bound for the Kolmogorov distance between distribution functions based on their Stieltjes transforms obtained in [11], Lemma 2.1. Let $\tilde{G}_y(x)$ denote the distribution function defined by the equality

$$\tilde{G}_y(x) = \frac{1 + \text{sign}(x) G_y(x^2)}{2}, \quad (3.1)$$

Recall that $G_y(x)$ is Marchenko–Pastur distribution function with parameter $y \in (0, 1]$. The distribution function $\tilde{G}_y(x)$ has a density

$$\tilde{G}'_y(x) = \frac{1}{2\pi|x|} \sqrt{(x^2 - a^2)(b^2 - x^2)} \mathbb{I}\{a \leq |x| \leq b\}. \quad (3.2)$$

For $y = 1$ the distribution function $\tilde{G}_y(x)$ is the distribution function of the semi-circular law. Given $\frac{\sqrt{y}}{2} \geq \varepsilon > 0$ introduce the interval $\mathbb{J}_\varepsilon = [1 - \sqrt{y} + \varepsilon, 1 + \sqrt{y} - \varepsilon]$ and $\mathbb{J}'_\varepsilon = [1 - \sqrt{y} + \frac{1}{2}\varepsilon, 1 + \sqrt{y} - \frac{1}{2}\varepsilon]$. For any x such that $|x| \in [1 - \sqrt{y}, 1 + \sqrt{y}]$, define $\gamma = \gamma(x) := \sqrt{y} - ||x| - 1|$. Note that $0 \leq \gamma \leq \sqrt{y}$. For any $x : |x| \in \mathbb{J}_\varepsilon$, we have $\gamma \geq \varepsilon$, respectively, for any $x : |x| \in \mathbb{J}'_\varepsilon$, we have $\gamma \geq \frac{1}{2}\varepsilon$. For a distribution function F denote by $S_F(z)$ its Stieltjes transform,

$$S_F(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x).$$

Proposition 3.1. *Let $v > 0$ and $H > 0$ and $\varepsilon > 0$ be positive numbers such that*

$$\tau = \frac{1}{\pi} \int_{|u| \leq H} \frac{1}{u^2 + 1} du = \frac{3}{4},$$

and

$$2vH \leq \varepsilon^{\frac{3}{2}}. \quad (3.3)$$

If \tilde{G}_y denotes the distribution function of the symmetrized (as in (3.1)) Marchenko–Pastur law, and F is any distribution function, there exist some absolute constants C_1, C_2, C_3 depending on y only such that

$$\begin{aligned} \Delta(F, \tilde{G}_y) &:= \sup_x |F(x) - \tilde{G}_y(x)| \\ &\leq 2 \sup_{x: |x| \in \mathbb{J}'_\varepsilon} \left| \operatorname{Im} \int_{-\infty}^x (S_F(u + i\frac{v}{\sqrt{\gamma}}) - S_{\tilde{G}_y}(u + i\frac{v}{\sqrt{\gamma}})) du \right| + C_1 v + C_2 \varepsilon^{\frac{3}{2}} \\ \text{with } C_1 &= \begin{cases} \frac{2H^2\sqrt{3}}{\pi^2\sqrt{y(1-\sqrt{y})}} & \text{if } 0 < y < 1, \\ \frac{H^2}{\pi} & \text{if } y = 1, \end{cases} \text{ and } C_2 = \begin{cases} \frac{4}{\pi\sqrt{y(1-\sqrt{y})}} & \text{if } 0 < y < 1, \\ \frac{1}{\pi} & \text{if } y = 1. \end{cases}. \end{aligned}$$

Remark 3.2.

$$H = \operatorname{tg} \frac{3\pi}{8} = 1 + \sqrt{2}.$$

Corollary 3.1. *Under the conditions of Proposition 3.1, for any $V > v$, the following inequality holds*

$$\begin{aligned} &\sup_{x \in \mathbb{J}'_\varepsilon} \left| \int_{-\infty}^x (\operatorname{Im}(S_F(u + iv') - S_{\tilde{G}_y}(u + iv')) du \right| \\ &\leq \int_{-\infty}^\infty |S_F(u + iV) - S_{\tilde{G}_y}(u + iV)| du \\ &\quad + \sup_{x \in \mathbb{J}'_\varepsilon} \left| \int_{v'}^V (S_F(x + iu) - S_{\tilde{G}_y}(x + iu)) du \right|. \end{aligned}$$

Proof. Let $x : |x| \in \mathbb{J}'_\varepsilon$ be fixed. Let $\gamma = \gamma(x) = \min\{|x| - 1 + \sqrt{y}, 1 + \sqrt{y} - |x|\}$. Set $z = u + iv'$ with $v' = \frac{v}{\sqrt{\gamma}}$, $v' \leq V$. Since the functions of $S_F(z)$ and $S_{\tilde{G}_y}(z)$ are analytic in the upper half-plane, it is enough to use Cauchy's theorem. We can write

$$\int_{-\infty}^x \operatorname{Im}(S_F(z) - S_{\tilde{G}_y}(z)) du = \lim_{L \rightarrow \infty} \int_{-L}^x (S_F(u + iv') - S_{\tilde{G}_y}(u + iv')) du,$$

for $x \in \mathcal{J}'_\varepsilon$. Since $v' = \frac{v}{\sqrt{\gamma}} \leq \frac{\varepsilon}{2H}$, without loss of generality we may assume that $v' \leq 2$. By Cauchy's integral formula, we have

$$\begin{aligned} \int_{-L}^x (S_F(z) - S_{\tilde{G}_y}(z)) du &= \int_{-L}^x (S_F(u + iV) - S_{\tilde{G}_y}(u + iV)) du \\ &\quad + \int_{v'}^V (S_F(-L + iu) - S_{\tilde{G}_y}(-L + iu)) du \\ &\quad - \int_{v'}^V (S_F(x + iu) - S_{\tilde{G}_y}(x + iu)) du. \end{aligned}$$

Denote by ξ (resp. η) a random variable with distribution function $F(x)$ (resp. $\tilde{G}_y(x)$). Then we have

$$|S_F(-L + iv')| = \left| \mathbf{E} \frac{1}{\xi + L - iv'} \right| \leq v'^{-1} \Pr\{|\xi| > L/2\} + \frac{2}{L}. \quad (3.4)$$

Similarly,

$$|S_{\tilde{G}_y}(-L + iv')| \leq v'^{-1} \Pr\{|\eta| > L/2\} + \frac{2}{L}.$$

These inequalities imply that

$$\left| \int_{v'}^V (S_F(-L + iu) - S_{\tilde{G}_y}(-L + iu)) du \right| \rightarrow 0 \quad \text{as } L \rightarrow \infty, \quad (3.5)$$

which completes the proof. \square

Combining the results of Proposition 3.1 and Corollary 3.1, we get

Corollary 3.2. *Under the conditions of Proposition 3.1 the following inequality holds*

$$\begin{aligned} \Delta(F, \tilde{G}_y) &\leq 2 \int_{-\infty}^{\infty} |S_F(u + iV) - S_{\tilde{G}_y}(u + iV)| du + C_1 v + C_2 \varepsilon^{\frac{3}{2}} \\ &\quad + 2 \sup_{x \in \mathbb{J}'_\varepsilon} \int_{v'}^V |S_F(x + iu) - S_{\tilde{G}_y}(x + iu)| du, \end{aligned} \quad (3.6)$$

where $v' = \frac{v}{\sqrt{\gamma}}$ with $\gamma = \min\{|x| - 1 + \sqrt{y}, 1 + \sqrt{y} - |x|\}$.

4 The proof of Theorem 1.1

We shall apply Corollary 3.2 to bound the Kolmogorov distance between the expected spectral distribution F_n and the Marchenko–Pastur distribution G_y . We denote the Stieltjes transform of $\mathcal{F}_n(x)$ by $m_n(z)$ and the Stieltjes transform of the Marchenko–Pastur law by $s_y(z)$. We shall use the “symmetrization” of the spectrum sample covariance matrix as in [12]. Introduce the $(p + n) \times (p + n)$ matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{O} & \mathbf{X} \\ \mathbf{X}^* & \mathbf{O} \end{bmatrix}, \quad (4.1)$$

where \mathbf{O} denotes a matrix with zero entries. Note that the eigenvalues of the matrix \mathbf{V} are $\pm s_1, \dots, \pm s_n$, and 0 with multiplicity $p - n$. Let $\mathbf{R} = \mathbf{R}(z)$ denote the resolvent matrix of \mathbf{V} defined by the equality

$$\mathbf{R} = (\mathbf{V} - z\mathbf{I}_{n+p})^{-1},$$

for all $z = u + iv$ with $v \neq 0$. Here and in what follows \mathbf{I}_k denotes the identity matrix of order k . Sometimes we shall omit the sub index in the notation of the identity matrix. If we consider the Stieltjes transforms $s_y(z)$ of the “symmetrized” Marchenko-Pastur distribution $\tilde{G}_y(x)$ (see formula (3.1)), then it is straightforward to check that $s_y(z) = zS_y(z^2)$ and

$$ys_y^2(z) + \left(\frac{y-1}{z} + z\right)s_y(z) + 1 = 0 \quad (4.2)$$

(see Section 3 in [12]). Furthermore, for the Stieltjes transform $\tilde{m}_n(z)$ of the “symmetrized” empirical spectral distribution function

$$\tilde{\mathcal{F}}_n(x) = \frac{1 + \text{sign}(x)\mathcal{F}_n(x^2)}{2}, \quad (4.3)$$

we have

$$\tilde{m}_n(z) = \frac{1}{n} \sum_{j=1}^n R_{jj} = \frac{1}{n} \sum_{j=n+1}^{n+p} R_{jj} + \frac{1-y}{yz}$$

(see, for instance, Section 3 in [12]). Note that by definition of a symmetrized distribution (4.3) we have

$$\sup_x |\mathcal{F}_n(x) - G_y(x)| = 2 \sup_x |\tilde{\mathcal{F}}_n(x) - \tilde{G}_y(x)|.$$

In what follows we shall consider symmetrized random values only. We shall omit the symbol “ \sim ” in the notation of the distribution function and its Stieltjes transform. Let $\mathbb{T}_j = \{1, \dots, n\} \setminus \{j\}$. For $j = 1, \dots, n$, introduce the matrices $\mathbf{V}^{(j)}$, obtained from \mathbf{V} by deleting the j -th row and j -th column, and define the corresponding resolvent matrix $\mathbf{R}^{(j)}$ by the equality $\mathbf{R}^{(j)} = (\mathbf{V}^{(j)} - z\mathbf{I}_{n+p-1})^{-1}$. Using the Schur decomposition formula we may show that

$$\mathbf{R} = \begin{bmatrix} z(\mathbf{X}\mathbf{X}^* - z^2\mathbf{I})^{-1} & \mathbf{X}(\mathbf{X}^*\mathbf{X} - z^2\mathbf{I})^{-1} \\ (\mathbf{X}^*\mathbf{X} - z^2\mathbf{I})^{-1}\mathbf{X}^* & z(\mathbf{X}^*\mathbf{X} - z^2\mathbf{I})^{-1} \end{bmatrix}. \quad (4.4)$$

From this representation it follows

$$\frac{1}{p} \sum_{k=1}^p R_{k+n, k+n} = ym_n(z) - \frac{1-y}{z}.$$

We shall use the representation, for $j = 1, \dots, n$,

$$R_{jj} = \frac{1}{-z - \frac{1}{p} \sum_{k,l=1}^p X_{jk} X_{jl} R_{k+n, l+n}^{(j)}} \quad (4.5)$$

(see, for example, Section 3 in [12]). We may rewrite it as follows

$$R_{jj} = -\frac{1}{z + ym_n(z) + \frac{y-1}{z}} + \frac{1}{z + ym_n(z) + \frac{y-1}{z}} \varepsilon_j R_{jj}, \quad (4.6)$$

where $\varepsilon_j = \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3}$ and

$$\begin{aligned} \varepsilon_{j1} &:= \frac{1}{p} \sum_{k=1}^p (X_{jk}^2 - 1) R_{k+n, k+n}^{(j)}, \quad \varepsilon_{j2} := \frac{1}{p} \sum_{1 \leq k \neq l \leq p} X_{jk} X_{jl} R_{k+n, l+n}^{(j)}, \\ \varepsilon_{j3} &:= \frac{1}{p} \left(\sum_{l=1}^p R_{l+n, l+n}^{(j)} - \sum_{l=1}^p R_{l+n, l+n} \right). \end{aligned}$$

We choose $V = 4\sqrt{y}$ and v_0 as defined in (1.14) and introduce the quantity $\varepsilon = (2av_0)^{\frac{2}{3}}$. We shall denote in what follows by C a generic constant depending on μ_4 and D only.

4.1 Estimation of the First Integral in (3.6) for $V = 4\sqrt{y}$

Denote by $\mathbb{T} = \{1, \dots, n\}$ and by $\mathbb{T}_{\mathbb{A}} = \mathbb{T} \setminus \mathbb{A}$ ($\mathbb{A} \subset \mathbb{T}$). In the following we shall systematically use for any $n \times p$ matrix \mathbf{X} together with its resolvent \mathbf{R} , its Stieltjes transform m_n etc. the corresponding quantities $\mathbf{X}^{(\mathbb{A})}$, $\mathbf{R}^{(\mathbb{A})}$ and $m_n^{(\mathbb{A})}$ for the corresponding sub matrix with entries $X_{jk}, j \in \mathbb{T}_{\mathbb{A}}, k = 1, \dots, p$. Observe that

$$m_n^{(\mathbb{A})}(z) = \frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{A}}} \frac{z}{(s_j^{(\mathbb{A})})^2 - z^2}. \quad (4.7)$$

By $\mathfrak{M}^{(\mathbb{A})}$ we denote the σ -algebra generated by X_{lk} with $l \in \mathbb{T}_{\mathbb{A}}, k = 1, \dots, p$. If $\mathbb{A} = \emptyset$ we shall omit the set \mathbb{A} as exponent index.

In this Section we shall consider $z = u + iV$ with $V = 4\sqrt{y}$. We shall use the representation (4.6).

Let

$$\Lambda_n := \Lambda_n(z) := m_n(z) - s_y(z) = \frac{1}{n} \sum_{j=1}^n R_{jj} - s_y(z).$$

It follows from (1.10), that for the symmetrized Marchenko – Pastur law we have

$$s_y(z) = -\frac{1}{z + \frac{y-1}{z} + ys_y(z)} \text{ and } |s_y(z)| \leq 1/\sqrt{y}. \quad (4.8)$$

See, for instance [13], Lemma 9.3. Summing the equalities (4.6) in $j = 1, \dots, n$ and solving with respect Λ_n , we get

$$\Lambda_n = m_n(z) - s_y(z) = \frac{T_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \quad (4.9)$$

where

$$T_n = \frac{1}{n} \sum_{j=1}^n \varepsilon_j R_{jj}.$$

Note that for $V = 4\sqrt{y}$

$$\begin{aligned} y \max\{|s_y(z)|, |m_n(z)|\} &\leq \frac{\sqrt{y}}{4} \leq \frac{1}{2} |z + y s_y(z) + \frac{y-1}{z}|, \\ y |s_y(z) - m_n(z)| &\leq \frac{\sqrt{y}}{2} \leq \frac{1}{2} |z + y s_y(z) + \frac{y-1}{z}| \text{ a.s.} \end{aligned} \quad (4.10)$$

This implies

$$\begin{aligned} |z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}| &\geq \frac{1}{2} |z + y s_y(z) + \frac{y-1}{z}| = \frac{1}{2 |s_y(z)|}, \\ |z + y m_n(z) + \frac{y-1}{z}| &\geq \frac{1}{2} |y s_y(z) + z + \frac{y-1}{z}|. \end{aligned} \quad (4.11)$$

The last inequalities and equality (4.9) imply as well that, for $V = 4\sqrt{y}$,

$$|m_n(z)| \leq |s_y(z)| (1 + 2|T_n(z)|). \quad (4.12)$$

Using equality (4.9), we may write,

$$\begin{aligned} \mathbf{E} \Lambda_n &= \mathbf{E} \frac{1}{n(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})} \sum_{j=1}^n (\varepsilon_{j1} + \varepsilon_{j2}) \\ &+ \mathbf{E} \frac{1}{n(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})} \sum_{j=1}^n \varepsilon_{j3} R_{jj} \\ &+ \mathbf{E} \frac{1}{n(z + m_n(z) + s_y(z) + \frac{y-1}{z})^2} \sum_{j=1}^n (\varepsilon_{j1} + \varepsilon_{j2}) \varepsilon_j R_{jj}. \end{aligned} \quad (4.13)$$

First we note that, by (4.4), for $\nu = 1, 2$

$$\begin{aligned} \mathbf{E} \frac{1}{n(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})} \sum_{j=1}^n \varepsilon_{j\nu} &= \sum_{j=1}^n \mathbf{E} \frac{1}{n(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})} \varepsilon_{j\nu} \\ &+ \mathbf{E} \frac{1}{n(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})} \sum_{j=1}^n \frac{\varepsilon_{j\nu} \varepsilon_{j3}}{z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}}. \end{aligned}$$

Observe that, for $\nu = 1, 2$,

$$\mathbf{E} \frac{1}{n(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})} \varepsilon_{j\nu} = 0. \quad (4.14)$$

Furthermore, using relations (4.8) and (4.10), we get

$$\begin{aligned} \left| \mathbf{E} \frac{1}{n(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})} \sum_{j=1}^n \frac{\varepsilon_{j\nu} \varepsilon_{j3}}{z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}} \right| \\ \leq \frac{C|s_y(z)|^2}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_{j\nu} \varepsilon_{j3}|. \end{aligned}$$

Applying Lemmas 8.12, 8.13, and 8.15, we conclude

$$\begin{aligned} \left| \mathbf{E} \frac{1}{n(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})} \sum_{j=1}^n \frac{\varepsilon_{j\nu} \varepsilon_{j3}}{z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}} \right| \\ \leq Cn^{-1} |s_y(z)|^2. \end{aligned} \quad (4.15)$$

We note now that, for $j = 1, \dots, n$

$$\sum_{l=1}^p R_{l+n, l+n} = nm_n(z) - \frac{p-n}{z}, \quad \sum_{l=1}^p R_{l+n, l+n}^{(j)} = nm_n^{(j)}(z) - \frac{p-n+1}{z}.$$

This implies that

$$\frac{1}{p} \left(\sum_{l=1}^p R_{l+n, l+n} - \sum_{l=1}^p R_{l+n, l+n}^{(j)} \right) = y(m_n(z) - m_n^{(j)}(z)) + \frac{1}{pz}. \quad (4.16)$$

On the other hand side

$$\mathrm{Tr} \mathbf{R} = 2nm_n(z) - \frac{p-n}{z}, \quad \mathrm{Tr} \mathbf{R}^{(j)} = 2nm_n^{(j)}(z) - \frac{p-n+1}{z}.$$

We get

$$m_n(z) - m_n^{(j)}(z) = \frac{1}{2n} (\mathrm{Tr} \mathbf{R} - \mathrm{Tr} \mathbf{R}^{(j)}) - \frac{1}{2nz}. \quad (4.17)$$

Comparing (4.16) and (4.17), we conclude

$$\varepsilon_{j3} = \frac{1}{p} \left(\sum_{l=1}^p R_{l+n, l+n} - \sum_{l=1}^p R_{l+n, l+n}^{(j)} \right) = \frac{y}{2n} (\mathrm{Tr} \mathbf{R} - \mathrm{Tr} \mathbf{R}^{(j)}) + \frac{y}{2nz}. \quad (4.18)$$

Multiply this equality by R_{jj} , summing in $j = 1, \dots, n$, and using that $(\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)})R_{jj} = -\frac{d}{dz}R_{jj}$, we get

$$\frac{1}{n} \sum_{j=1}^n \varepsilon_{jj} R_{jj} = -\frac{d}{dz} \left(\frac{y}{n} \sum_{j=1}^n R_{jj} \right) + \frac{y}{2nz} \frac{1}{n} \sum_{j=1}^n R_{jj} = -y \frac{d}{dz} m_n(z) + \frac{y}{2nz} m_n(z). \quad (4.19)$$

Continuing with

$$\left| \frac{d}{dz} m_n(z) \right| = \left| \frac{d}{dz} \left(\frac{1}{2n} \text{Tr } \mathbf{R} + \frac{p-n}{2nz} \right) \right| = \left| \frac{1}{2n} \text{Tr } \mathbf{R}^2 - \frac{p-n}{2nz^2} \right|$$

and

$$|\text{Tr } \mathbf{R}^2| \leq v^{-1} \text{Im}(\text{Tr } \mathbf{R}),$$

we get

$$\left| \frac{d}{dz} m_n(z) \right| \leq \frac{C}{v} \text{Im } m_n(z) + \frac{C(1-y)}{|z|^2}.$$

The last inequality and inequality (4.11) together imply, for $V = 4\sqrt{y}$,

$$\left| \frac{1}{n(z + m_n(z) + s_y(z) + \frac{y-1}{z})} \sum_{j=1}^n \varepsilon_{j3} R_{jj} \right| \leq \frac{C|s_y(z)|^2}{n} + \frac{C(1-y)|s_y(z)|}{n|z|^2}.$$

Applying inequalities (4.11), we get, for $V = 4\sqrt{y}$,

$$\begin{aligned} \left| \mathbf{E} \frac{1}{n(z + m_n(z) + s_y(z) + \frac{y-1}{z})} \sum_{j=1}^n \frac{\varepsilon_{j\nu} \varepsilon_{j3}}{z + m_n(z) + s_y(z) + \frac{y-1}{z}} \right| \\ \leq \frac{C|s_y(z)|^2}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_{j\nu} \varepsilon_j|. \end{aligned}$$

According to Lemmas 8.12, 8.13, and 8.15, we obtain

$$\left| \mathbf{E} \frac{1}{n(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})} \sum_{j=1}^n \frac{\varepsilon_{j\nu} \varepsilon_{j3}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right| \leq \frac{C|s_y(z)|^2}{n}. \quad (4.20)$$

Combining now inequalities (4.15), (4.19), (4.13), and relations (4.13), (4.14), we conclude

$$|\mathbf{E} \Lambda_n| \leq \frac{C|s_y(z)|^2}{n} + \frac{C|s_y(z)|}{n|z|^2}.$$

Furthermore

$$\int_{-\infty}^{\infty} |s_y(u + iV)|^2 du \leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{(x - u)^2 + V^2} du \right) dG_y(x) \leq \frac{1}{2\pi V},$$

and

$$\int_{-\infty}^{\infty} \frac{du}{u^2 + V^2} \leq \frac{1}{\pi V}.$$

These inequalities imply that

$$\int_{-\infty}^{\infty} |\mathbf{E}\Lambda_n(u + iV)| du \leq \frac{C}{n}. \quad (4.21)$$

4.2 The Bound of the Second Integral in (3.6)

To finish the proof of Theorem 1.1 we need to bound the second integral in (3.2) for $z \in \mathbb{G}$, $v_0 = C_7 n^{-1}$ and $\varepsilon = C_8 v_0^{\frac{2}{3}}$, where the constant C_8 is chosen such that so that condition (3.3) holds. We shall use the results of Theorem 1.3. According to these results we have, for $z \in \mathbb{G}$,

$$|\mathbf{E}\Lambda_n| \leq \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}. \quad (4.22)$$

We have

$$\int_{v_0/\sqrt{\gamma}}^V |\mathbf{E}(m_n(x + iv) - s(x + iv))| dv \leq \frac{1}{n} \int_{\frac{v_0}{\sqrt{\gamma}}}^V \frac{dv}{v^{\frac{3}{4}}} + \frac{1}{n\sqrt{n}\gamma^{\frac{1}{4}}} \int_{\frac{v_0}{\sqrt{\gamma}}}^V \frac{dv}{v^{\frac{3}{2}}}.$$

After integrating we get

$$\int_{v_0/\sqrt{\gamma}}^V |\mathbf{E}(m_n(x + iv) - s_y(x + iv))| dv \leq \frac{C}{n} + \frac{C\gamma^{\frac{1}{4}}}{n\sqrt{n}\gamma^{\frac{1}{4}}v_0^{\frac{1}{2}}} \leq \frac{C}{n}. \quad (4.23)$$

Inequalities (4.21) and (4.23) complete the proof of Theorem 1.1. Thus Theorem 1.1 is proved.

5 Proof of Corollary 1.1

To prove the Corollary 1.1 we consider truncated random variables \widehat{X}_{jl} defined by

$$\widehat{X}_{jl} := X_{jl} \mathbb{I}\{|X_{jl}| \leq cn^{\frac{1}{4}}\} \quad (5.1)$$

and introduce matrices

$$\widehat{\mathbf{X}} = \frac{1}{\sqrt{p}}(\widehat{X}_{jl}), \quad \widehat{\mathbf{V}} = \begin{bmatrix} \mathbf{O} & \widehat{\mathbf{X}} \\ (\widehat{\mathbf{X}})^* & \mathbf{O} \end{bmatrix}.$$

Let $\widehat{\mathcal{F}}_n(x)$ denote the empirical spectral distribution function of the matrix $\widehat{\mathbf{V}}$.

Lemma 5.1. *Assuming the conditions of Theorem 1.1 there exists a constant $C > 0$ depending on μ_8 only such that*

$$\mathbf{E}\{\sup_x |\mathcal{F}_n(z) - \widehat{\mathcal{F}}_n(x)|\} \leq \frac{C}{n}.$$

Proof. We shall use the rank inequality of Bai. See [3], Theorem A.43, p. 503. According this inequality

$$\mathbf{E}\{\sup_x |\mathcal{F}_n(x) - \widehat{\mathcal{F}}_n(x)|\} \leq \frac{2}{n} \mathbf{E}\{\text{rank}(\mathbf{X} - \widehat{\mathbf{X}})\}.$$

Observing that the rank of a matrix is not larger then numbers of its non-zero entries, we may write

$$\begin{aligned} \mathbf{E}\{\sup_x |\mathcal{F}_n(x) - \widehat{\mathcal{F}}_n(x)|\} &\leq \frac{2}{n} \sum_{j=1}^n \sum_{k=1}^p \mathbf{E}\mathbb{I}\{|X_{jk}| \geq Cn^{\frac{1}{4}}\} \\ &\leq \frac{1}{n^3} \sum_{j,k=1}^n \mathbf{E}|X_{jk}|^8 \leq \frac{C\mu_8}{n}. \end{aligned}$$

Thus, the Lemma is proved. \square

We shall compare the Stieltjes transform of the matrix $\widehat{\mathbf{V}}$ and the matrix obtained from $\widehat{\mathbf{V}}$ by centralizing and normalizing its entries. Introduce $\widetilde{X}_{jk} = \widehat{X}_{jk} - \mathbf{E}\widehat{X}_{jk}$ and $\widetilde{\mathbf{X}} = \frac{1}{\sqrt{p}}(\widetilde{X}_{jk})_{j,k=1}^n$. We normalize the r.v.'s \widetilde{X}_{jk} . Let $\sigma_{jk}^2 = \mathbf{E}|\widetilde{X}_{jk}|^2$. We define the r.v.'s $\check{X}_{jk} = \sigma_{jk}^{-1} \widetilde{X}_{jk}$. Let $\check{\mathbf{X}} = \frac{1}{\sqrt{p}}(\check{X}_{jk})_{j,k=1}^n$. Finally, let $\check{m}_n(z)$ denote Stieltjes transform of empirical spectral distribution function of the matrix $\check{\mathbf{V}} = \begin{bmatrix} \mathbf{O} & \check{\mathbf{X}} \\ \check{\mathbf{X}}^* & \mathbf{O} \end{bmatrix}$.

Remark 5.1. *Note that*

$$|\check{X}_{jl}| \leq D_1 n^{\frac{1}{4}}, \quad \mathbf{E}\check{X}_{jl} = 0 \text{ and } \mathbf{E}\check{X}_{jk}^2 = 1, \quad (5.2)$$

for some absolute constant D_1 . That means that the matrix $\check{\mathbf{X}}$ satisfies the conditions of Theorem 1.3.

Lemma 5.2. *There exists some absolute constant C depending on μ_8 such that*

$$|\mathbf{E}(\tilde{m}_n(z) - \check{m}_n(z))| \leq \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}}.$$

Proof. Note that

$$\begin{aligned} \check{m}_n(z) &= \frac{1}{2n} \text{Tr}(\check{\mathbf{V}} - z\mathbf{I})^{-1} + \frac{p-n}{2nz} =: \frac{1}{2n} \text{Tr} \check{\mathbf{R}} + \frac{p-n}{2nz}, \\ \tilde{m}_n(z) &= \frac{1}{2n} \text{Tr}(\tilde{\mathbf{V}} - z\mathbf{I})^{-1} + \frac{p-n}{2nz} =: \frac{1}{2n} \text{Tr} \tilde{\mathbf{R}} + \frac{p-n}{2nz}. \end{aligned}$$

Therefore,

$$\tilde{m}_n(z) - \check{m}_n(z) = \frac{1}{2n} \text{Tr}(\tilde{\mathbf{R}} - \check{\mathbf{R}}) = \frac{1}{n} \text{Tr}(\tilde{\mathbf{V}} - \check{\mathbf{V}}) \tilde{\mathbf{R}} \check{\mathbf{R}}. \quad (5.3)$$

Using the simple inequalities $|\text{Tr} \mathbf{AB}| \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2$ and $\|\mathbf{AB}\|_2 \leq \|\mathbf{A}\| \|\mathbf{B}\|_2$, we get

$$|\mathbf{E}(\tilde{m}_n(z) - \check{m}_n(z))| \leq n^{-1} \mathbf{E}^{\frac{1}{2}} \|\tilde{\mathbf{R}}\|^2 \|\check{\mathbf{R}}\|_2^2 \mathbf{E}^{\frac{1}{2}} \|\tilde{\mathbf{V}} - \check{\mathbf{V}}\|_2^2. \quad (5.4)$$

Furthermore, we note that,

$$\tilde{\mathbf{V}} - \check{\mathbf{V}} = \frac{1}{\sqrt{p}} \begin{bmatrix} \mathbf{O} & \tilde{\mathbf{X}} - \check{\mathbf{X}} \\ (\tilde{\mathbf{X}} - \check{\mathbf{X}})^* & \mathbf{O} \end{bmatrix} \quad (5.5)$$

and

$$\|\tilde{\mathbf{V}} - \check{\mathbf{V}}\|_2 \leq 2 \max_{1 \leq j, k \leq n} \{1 - \sigma_{jk}\} \|\tilde{\mathbf{X}}\|_2.$$

Since

$$0 < 1 - \sigma_{jk} \leq 1 - \sigma_{jk}^2 \leq Cn^{-\frac{3}{2}}\mu_8,$$

therefore

$$\mathbf{E} \|\tilde{\mathbf{V}} - \check{\mathbf{V}}\|_2^2 \leq C\mu_8^2 n^{-2}. \quad (5.6)$$

Applying Lemma 8.4, inequality (8.6), in the Appendix and inequality (5.6), we obtain

$$|\mathbf{E}(\tilde{m}_n(z) - \check{m}_n(z))| \leq Cn^{-\frac{3}{2}}v^{-\frac{3}{2}} \left(\frac{1}{n} \sum_{j=1}^n \mathbf{E} |\check{\mathbf{R}}_{jj}| \right)^{\frac{1}{2}}.$$

According Remark 5.1, we may apply Corollary 6.15 in Section 6.2 with $q = 1$ to prove the claim. Thus, Lemma 5.2 is proved. \square

Denote by $\tilde{m}_n(z)$ the Stieltjes transform of the empirical distribution function of the matrix $\tilde{\mathbf{V}}$ and let $\hat{m}_n(z)$ denote the Stieltjes transform of the matrix $\hat{\mathbf{V}}$.

Lemma 5.3. *For some absolute constant $C > 0$ we have*

$$|\mathbf{E}(\tilde{m}_n(z) - \hat{m}_n(z))| \leq \frac{C\mu_8}{n^{\frac{3}{2}}v^{\frac{3}{2}}}.$$

Proof. Similar to (5.3), we write

$$\tilde{m}_n(z) - \hat{m}_n(z) = \frac{1}{n} \text{Tr}(\tilde{\mathbf{R}} - \hat{\mathbf{R}}) = \frac{1}{n} \text{Tr}(\tilde{\mathbf{V}} - \hat{\mathbf{V}}) \tilde{\mathbf{R}} \hat{\mathbf{R}}.$$

This yields

$$\mathbf{E}|\tilde{m}_n(z) - \hat{m}_n(z)| \leq n^{-1} \mathbf{E} \|\hat{\mathbf{R}}\| \|\tilde{\mathbf{R}}\|_2 \|\mathbf{E}\hat{\mathbf{V}}\|_2. \quad (5.7)$$

Furthermore, we note that, by definition (5.1) and condition (1.5), we have

$$|\mathbf{E}\hat{X}_{jk}| \leq Cn^{-\frac{7}{4}}\mu_8. \quad (5.8)$$

Applying Lemma 8.4, inequality (8.6), in the Appendix and inequality (5.8), we obtain using $\|\hat{\mathbf{R}}\| \leq v^{-1}$,

$$\mathbf{E}|\tilde{m}_n(z) - \hat{m}_n(z)| \leq n^{-\frac{7}{4}}v^{-\frac{3}{2}}\mathbf{E}^{\frac{1}{2}}|\tilde{m}_n(z)|.$$

By Lemma 5.2,

$$\mathbf{E}|\tilde{m}_n(z)| \leq \mathbf{E}|\check{m}_n(z)| + C,$$

for some constant C depending on μ_8 and A_0 . According to Corollary 6.15 in Section 6.2 with $q = 1$

$$\mathbf{E}|\check{m}_n(z)| \leq \frac{1}{n} \sum_{j=1}^n \mathbf{E}|\check{R}_{jj}| \leq C,$$

with a constant C depending on μ_4 , D . Using these inequalities, we get

$$|\mathbf{E}\tilde{m}_n(z) - \hat{m}_n(z)| \leq \frac{C\mu_8}{n^{\frac{7}{4}}v^{\frac{3}{2}}} \leq \frac{C\mu_8}{n^{\frac{3}{2}}v^{\frac{3}{2}}}.$$

Thus Lemma 5.3 is proved. \square

Corollary 5.4. *Assuming the conditions of Corollary 1.1, we have for $z \in \mathbb{G}$,*

$$|\mathbf{E}\hat{m}_n(z) - s_y(z)| \leq \frac{C}{(nv)^{\frac{3}{2}}} + \frac{C}{n^2v^2\sqrt{\gamma}}.$$

Proof. The proof immediately follows from the inequality

$$|\mathbf{E}\widehat{m}_n(z) - s_y(z)| \leq |\mathbf{E}(\widehat{m}_n(z) - \check{m}_n(z))| + |\mathbf{E}\check{m}_n(z) - s(z)|,$$

Lemmas 5.2 and 5.3 and Theorem 1.3. \square

The proof of Corollary 1.1 follows now from Lemma 5.1, Corollary 3.2, inequality (4.21) and inequality

$$\sup_{x \in \mathbb{J}_\varepsilon} \int_{v_0/\sqrt{\gamma}}^V |\mathbf{E}\widehat{m}_n(x + iv) - s(x + iv)| dv \leq \frac{C}{n}.$$

6 The Crucial Results

The main problem in proving Theorem 1.3 is the the derivation of the following bound

$$\mathbf{E}|R_{jj}|^q \leq C^q,$$

for $j = 1, \dots, n$ and any $z \in \mathbb{G}$. This bound in the case of Wigner matrices was shown in [15]. To prove this bound we used an approach similar to that of Lemma 3.4 in [20]. We succeeded in the case of finite moments only developing new bounds of quadratic forms of the following type

$$\mathbf{E} \left| \frac{1}{n} \sum_{1 \leq l \neq k \leq p} X_{jl} X_{jk} R_{k+n, l+n}^{(j)} \right|^q \leq \left(\frac{Cq}{\sqrt{nv}} \right)^q.$$

These estimates are based on a recursive scheme of using Rosenthal's and Burkholder's inequalities.

6.1 The Key Lemma

In this Section we provide auxiliary lemmas needed for the proof of Theorem 1.1. Recall that the Stieltjes transform of an empirical spectral distribution function $\mathcal{F}_n(x)$, say $m_n(z)$, is given by

$$m_n(z) = \frac{1}{n} \sum_{j=1}^n R_{jj} = \frac{1}{2n} \text{Tr } \mathbf{R} + \frac{p-n}{2nz}. \quad (6.1)$$

For any $\mathbb{J} \subset \mathbb{T} = \{1, \dots, n\}$ denote $\mathbb{T}_{\mathbb{J}} = \mathbb{T} \setminus \mathbb{J}$. For any $\mathbb{J} \subset \mathbb{T}$ and $j \in \mathbb{T}_{\mathbb{J}}$ define the quadratic form,

$$Q^{(\mathbb{J}, j)} := \frac{1}{p} \sum_{l=1}^{p-1} \left| \sum_{k=1}^{l-1} X_{jk} R_{k+n, l+n}^{(\mathbb{J}, j)} \right|^2.$$

Similar, for $\widehat{\mathbb{J}} \subset \widehat{\mathbb{T}} = \{1, \dots, p\}$ and $j \in \widehat{\mathbb{T}}_{\widehat{\mathbb{J}}}$,

$$Q^{(\widehat{\mathbb{J}}, j)} := \frac{1}{p} \sum_{l=1}^{n-1} \left| \sum_{k=1}^{l-1} X_{kj} R_{kl}^{(\widehat{\mathbb{J}}+n, j+n)} \right|^2.$$

Theorem 6.1. *Assuming the conditions of Theorem 1.1 there exist constants A_1, C, C_3 depending on μ_4 and D only such that we have for $v \geq v_0$ and $q \leq A_1(nv)^{\frac{1}{4}}$ and for any $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq C \log n$,*

$$\mathbf{E}(Q^{(\mathbb{J}, j)})^q \leq (C_3 q)^{2q} v^{-q}. \quad (6.2)$$

Respectively, for any $\widehat{\mathbb{J}} \subset \widehat{\mathbb{T}}$,

$$\mathbf{E}(Q^{(\widehat{\mathbb{J}}, j)})^q \leq (C_3 q)^{2q} v^{-q}. \quad (6.3)$$

Corollary 6.1. *Assuming the conditions of Theorem 1.1 and for $z = u + iV$ with $V = 4\sqrt{y}$, we have*

$$\mathbf{E}(Q^{(\mathbb{J}, j)})^q \leq C^q q^{2q},$$

and

$$\mathbf{E}(Q^{(\widehat{\mathbb{J}}, j)})^q \leq C^q q^{2q}.$$

Proof. The result immediately follows from Theorem 6.1 □

Proof of Theorem 6.1. We proof inequality (6.2) only. The inequality (6.3) is proved similar. For the proof of Theorem 6.1 we prove several auxiliary Lemmas.

For any $\mathbb{J} \subset \mathbb{T}$ introduce $\mathbb{T}_{\mathbb{J}} = \mathbb{T} \setminus \mathbb{J}$. We introduce the quantity, for some $\mathbb{J} \subset \mathbb{T}$,

$$B_q^{(\mathbb{J})} := \left[\frac{1}{p} \sum_{l=2}^p \left(\sum_{k=1}^{l-1} |R_{k+n, l+n}^{(\mathbb{J})}|^2 \right)^q \right].$$

By Lemma 8.4, inequality (8.8) in the Appendix, we have

$$\mathbf{E} B_q^{(\mathbb{J})} \leq v^{-q} \frac{1}{n} \sum_{l=1}^p \mathbf{E} |R_{l+n, l+n}^{(\mathbb{J})}|^q. \quad (6.4)$$

Furthermore, introduce the quantities

$$\begin{aligned}
Q_{\nu}^{(\mathbb{J},j)} &= \sum_{l=2}^p \left| \sum_{k=1}^{l-1} X_{jk} a_{lk}^{(\mathbb{J},j,\nu)} \right|^2, \\
Q_{\nu 1}^{(\mathbb{J},j)} &= \sum_{l=2}^p a_{ll}^{(\mathbb{J},j,\nu+1)}, \\
Q_{\nu 2}^{(\mathbb{J},j)} &= \sum_{l=2}^p (X_{jl}^2 - 1) a_{ll}^{(\mathbb{J},j,\nu+1)}, \\
Q_{\nu 3}^{(\mathbb{J},j)} &= \sum_{1 \leq l \neq k \leq p} X_{jk} X_{jl} a_{kl}^{(\mathbb{J},j,\nu+1)},
\end{aligned} \tag{6.5}$$

where, $a_{kl}^{(\mathbb{J},j,\nu)}$ are defined recursively via

$$\begin{aligned}
a_{kl}^{(\mathbb{J},j,0)} &= \frac{1}{\sqrt{p}} R_{k+n,l+n}^{(\mathbb{J},j)}, \\
a_{kl}^{(\mathbb{J},j,\nu+1)} &= \sum_{r=k \vee l + 1}^p a_{rl}^{(\mathbb{J},j,\nu)} \overline{a_{rk}^{(\mathbb{J},j,\nu)}}, \text{ for } \nu = 0, \dots, L-1,
\end{aligned} \tag{6.6}$$

where $k \vee l := \max\{k, l\}$. Using these notations we have

$$Q_{\nu}^{(\mathbb{J},k)} = Q_{\nu 1}^{(\mathbb{J},k)} + Q_{\nu 2}^{(\mathbb{J},k)} + Q_{\nu 3}^{(\mathbb{J},k)}. \tag{6.7}$$

Lemma 6.2. *Under the conditions of Theorem 1.1 we have*

$$\sum_{l=1}^p |a_{kl}^{(\mathbb{J},j,\nu+1)}|^2 \leq \left(\sum_{r=1}^p |a_{kr}^{(\mathbb{J},j,\nu)}|^2 \right) \left(\sum_{l,r=1}^p |a_{lr}^{(\mathbb{J},j,\nu)}|^2 \right). \tag{6.8}$$

Moreover,

$$\sum_{l,k=1}^p |a_{kl}^{(\mathbb{J},j,\nu+1)}|^2 \leq \left(\sum_{k,l=1}^p |a_{kl}^{(\mathbb{J},j,\nu)}|^2 \right)^2. \tag{6.9}$$

Proof. We apply Cauchy – Schwartz inequality and obtain

$$|a_{l,k}^{(\mathbb{J},j,\nu+1)}|^2 \leq \sum_{r=k \vee l + 1}^p |a_{rl}^{(\mathbb{J},j,\nu)}|^2 \sum_{r=k \vee l + 1}^p |a_{kr}^{(\mathbb{J},j,\nu)}|^2.$$

Summing in k and l , (6.8) and (6.9) follow. □

Corollary 6.3. *Under the conditions of Theorem 1.1 we have*

$$\sum_{k,l=1}^p |a_{kl}^{(\mathbb{J},j,\nu)}|^2 \leq \left(\operatorname{Im} \left(m_n^{(\mathbb{J},j)}(z) - \frac{p-n+|\mathbb{J}|}{nz} \right) v^{-1} \right)^{2^\nu} \quad (6.10)$$

and

$$\sum_{l=1}^p |a_{kl}^{(\mathbb{J},j,\nu)}|^2 \leq \left(\operatorname{Im} \left(m_n^{(\mathbb{J},j)}(z) - \frac{p-n+|\mathbb{J}|}{nz} \right) v^{-1} \right)^{2^\nu-1} n^{-1} v^{-1} \operatorname{Im} R_{k+n,k+n}^{(\mathbb{J},j)}.$$

Proof. By definition of $a_{kl}^{(\mathbb{J},j,0)}$, see (6.6), applying Lemma 8.4, equality (8.6), in the Appendix, we get

$$\sum_{l=1}^p |a_{kl}^{(\mathbb{J},j,0)}|^2 \leq \frac{1}{p} \sum_{l=1}^p |R_{k+n,l+n}^{(\mathbb{J},j)}|^2 \leq n^{-1} v^{-1} \left(\operatorname{Im} R_{k+n,k+n}^{(\mathbb{J},j)} \right),$$

The general case follows now by induction in ν , Lemma 6.2, and inequality (8.6), Lemma 8.4 in the Appendix. \square

Corollary 6.4. *Under the conditions of Theorem 1.1 we have*

$$a_{ll}^{(\mathbb{J},j,\nu+1)} \leq (nv)^{-1} \left(v^{-1} \operatorname{Im} \left(m_n^{(\mathbb{J},j)}(z) - \frac{p-n+|\mathbb{J}|}{nz} \right) \right)^{2^\nu-1} \operatorname{Im} R_{l+n,l+n}^{(\mathbb{J},j)}.$$

Proof. The result immediately follows from the definition of $a_{ll}^{(\mathbb{J},j,\nu)}$ and Corollary 6.3. \square

Corollary 6.5. *Under the conditions of Theorem 1.1 we have*

$$\frac{1}{n^2} \sum_{l,k=1}^p |a_{lk}^{(\mathbb{J},\nu+1)}|^q \leq (nv)^{-q} \left(v^{-1} \operatorname{Im} \left(m_n^{(\mathbb{J},j)}(z) - \frac{p-n+|\mathbb{J}|}{nz} \right) \right)^{(2^\nu-1)q} \left(\frac{1}{n} \sum_{k=1}^p |R_{k+n,k+n}^{(\mathbb{J},j)}|^{\frac{q}{2}} \right)^2$$

and

$$\frac{1}{n} \sum_{l=1}^p \left(\sum_{k=1}^p |a_{lk}^{(\mathbb{J},\nu+1)}|^2 \right)^{\frac{q}{2}} \leq (nv)^{-\frac{q}{2}} \left(v^{-1} \operatorname{Im} \left(m_n^{(\mathbb{J},j)}(z) - \frac{p-n+|\mathbb{J}|}{nz} \right) \right)^{(2^\nu-1)\frac{q}{2}} \frac{1}{n} \sum_{k=1}^p |R_{k+n,k+n}^{(\mathbb{J},j)}|^{\frac{q}{2}}$$

Proof. By definition of $a_{lk}^{(\mathbb{J},\nu+1)}$ and Cauchy's inequality, we have

$$|a_{lk}^{(\mathbb{J},\nu+1)}|^q \leq \left(\sum_{r=1}^p |a_{lr}^{(\mathbb{J},\nu)}|^2 \right)^{\frac{q}{2}} \left(\sum_{r=1}^p |a_{rk}^{(\mathbb{J},\nu)}|^2 \right)^{\frac{q}{2}}. \quad (6.11)$$

Using Corollary 6.3 and summing in l, k , we get

$$\frac{1}{n^2} \sum_{l,k=1}^p |a_{lk}^{(\mathbb{J}, \nu+1)}|^q \leq n^{-q} v^{-(2\nu-1)q} \text{Im}^{(2\nu-1)q} \left(m_n^{(\mathbb{J})}(z) - \frac{p-n+|\mathbb{J}|}{nz} \right) \left(\frac{1}{n} \sum_{k=1}^p |R_{k+n,k+n}^{(\mathbb{J},j)}|^{\frac{q}{2}} \right)^2. \quad (6.12)$$

□

In what follows we shall use the notations

$$\begin{aligned} \Psi^{(\mathbb{J})} &:= \text{Im} \left(m_n^{(\mathbb{J})}(z) - \frac{p-n+|\mathbb{J}|}{nz} \right) + \frac{1}{nv}, \quad (A_{\nu,q}^{(\mathbb{J})})^2 = \mathbf{E}(\Psi^{(\mathbb{J})})^{(2\nu-1)2q}, \\ T_{\nu,q}^{(\mathbb{J},j)} &:= \mathbf{E}|Q_{\nu}^{(\mathbb{J},j)}|^q, \quad A_q^{(\mathbb{J})} := 1 + \mathbf{E}^{\frac{1}{4}}|\Psi^{(\mathbb{J})}|^{4q}. \end{aligned} \quad (6.13)$$

Note that

$$\text{Im} \left(m_n^{(\mathbb{J},j)}(z) - \frac{p-n+|\mathbb{J}|}{nz} \right) \leq \Psi^{(\mathbb{J})} \quad (6.14)$$

Let s_0 denote some fixed number (for instance $s_0 = 2^4$). Let A_1 be a constant (to be chosen later) and $0 < v_1 \leq 4$ a constant such that $v_0 = A_0 n^{-1} \leq v_1$ for all $n \geq 1$.

Lemma 6.6. *Assuming the conditions of Theorem 1.1 and for $q \leq A_1(nv)^{\frac{1}{4}}$*

$$\mathbf{E}|R_{l+n,l+n}^{(\mathbb{J})}|^q \leq C_0^q, \text{ for } v \geq v_1, \text{ for all } l = 1, \dots, p, \quad (6.15)$$

we have for $v \geq v_1/s_0$ and $q \leq A_1(nv)^{\frac{1}{4}}$, and $j \in \mathbb{T}_{\mathbb{J}}$

$$\mathbf{E}(Q_0^{(\mathbb{J},j)})^q \leq 6 \left(\frac{C_3 q}{\sqrt{2}} \right)^{2q} v^{-q} A_q^{(\mathbb{J})}, \quad (6.16)$$

where C_3 is a constant depending on C_0 .

Proof. Using the representation (6.7) and the triangle inequality, we get

$$\mathbf{E}|Q_{\nu}^{(\mathbb{J},j)}|^q \leq 3^q \left(\mathbf{E}|Q_{\nu 1}^{(\mathbb{J},j)}|^q + \mathbf{E}|Q_{\nu 2}^{(\mathbb{J},j)}|^q + \mathbf{E}|Q_{\nu 3}^{(\mathbb{J},j)}|^q \right). \quad (6.17)$$

Let $\mathfrak{M}^{(\mathbb{A})}$ denote the σ -algebra generated by r.v.'s $X_{j,l}$ for $j \in \mathbb{T}_{\mathbb{A}}$, $l = 1, \dots, p$, for any set \mathbb{A} . Conditioning on $\mathfrak{M}^{(\mathbb{J},j)}$ ($\mathbb{A} = \mathbb{J} \cup \{j\}$) and applying Rosenthal's inequality (see Lemma 8.1), we get

$$\mathbf{E}|Q_{\nu 2}^{(\mathbb{J},j)}|^q \leq C_1^q q^q \left(\mathbf{E} \left(\sum_{l=1}^p |a_{ll}^{(\mathbb{J},j,\nu+1)}|^2 \right)^{\frac{q}{2}} + \sum_{l=1}^p \mathbf{E}|a_{ll}^{(\mathbb{J},j,\nu+1)}|^q \mathbf{E}|X_{jl}|^{2q} \right),$$

where C_1 denotes the absolute constant in Rosenthal's inequality. By Remark 5.1, we get

$$\mathbf{E}|Q_{\nu 2}^{(\mathbb{J}, j)}|^q \leq C_1^q q^q \left(\mathbf{E} \left(\sum_{l=1}^p |a_{ll}^{(\mathbb{J}, j, \nu+1)}|^2 \right)^{\frac{q}{2}} + p^{\frac{q}{2}} \frac{1}{p} \sum_{l=1}^p \mathbf{E} |a_{ll}^{(\mathbb{J}, j, \nu+1)}|^q \right). \quad (6.18)$$

Analogously conditioning on $\mathfrak{M}^{(\mathbb{J}, j)}$ and applying Burkholder's inequality (see Lemma 8.3), we get

$$\begin{aligned} \mathbf{E}|Q_{\nu 3}^{(\mathbb{J}, j)}|^q &\leq C_2^q q^q \left(\mathbf{E} \left(\sum_{l=2}^p \left| \sum_{k=1}^{l-1} X_{jk} a_{kl}^{(\mathbb{J}, j, \nu+1)} \right|^2 \right)^{\frac{q}{2}} \right. \\ &\quad \left. + \sum_{l=1}^{p-1} \mathbf{E} \left| \sum_{k=1}^{l-1} X_{jk} a_{lk}^{(\mathbb{J}, j, \nu+1)} \right|^q \mathbf{E}|X_{jl}|^q \right), \end{aligned} \quad (6.19)$$

where C_2 denotes the absolute constant in Burkholder's inequality. Conditioning again on $\mathfrak{M}^{(\mathbb{J}, j)}$ and applying Rosenthal's inequality, we obtain

$$\begin{aligned} \mathbf{E} \left| \sum_{l=1}^p X_{jl} a_{kl}^{(\mathbb{J}, j, \nu+1)} \right|^q &\leq C_1^q q^q \left(\mathbf{E} \left(\sum_{l=1}^p |a_{lk}^{(\mathbb{J}, j, \nu+1)}|^2 \right)^{\frac{q}{2}} \right. \\ &\quad \left. + \sum_{l=1}^p \mathbf{E} |a_{lk}^{(\mathbb{J}, j, \nu+1)}|^q \mathbf{E}|X_{jl}|^q \right). \end{aligned} \quad (6.20)$$

Combining inequalities (6.19) and (6.20), we get

$$\begin{aligned} \mathbf{E}|Q_{\nu 3}^{(\mathbb{J}, j)}|^q &\leq C_2^q q^q \mathbf{E}|Q_{\nu+1}^{(\mathbb{J}, j)}|^{\frac{q}{2}} + C_1^q C_2^q q^{2q} \sum_{l=1}^p \mathbf{E} \left(\sum_{k=1}^p |a_{kl}^{(\mathbb{J}, j, \nu+1)}|^2 \right)^{\frac{q}{2}} \mathbf{E}|X_{jk}|^q \\ &\quad + C_1^q C_2^q q^{2q} \sum_{l=1}^p \sum_{k=1}^p \mathbf{E} |a_{kl}^{(\mathbb{J}, j, \nu+1)}|^q \mathbf{E}|X_{jk}|^q \mathbf{E}|X_{jl}|^q. \end{aligned}$$

Using Remark 5.1, this implies

$$\begin{aligned} \mathbf{E}|Q_{\nu 3}^{(\mathbb{J}, j)}|^q &\leq C_2^q q^q \mathbf{E}|Q_{\nu+1}^{(\mathbb{J}, j)}|^{\frac{q}{2}} + C_1^q C_2^q q^{2q} n^{\frac{q}{4}} \frac{1}{p} \sum_{l=1}^p \mathbf{E} \left(\sum_{k=1}^p |a_{lk}^{(\mathbb{J}, j, \nu+1)}|^2 \right)^{\frac{q}{2}} \\ &\quad + C_1^q C_2^q q^{2q} n^{\frac{q}{2}} \frac{1}{n^2} \sum_{l=1}^p \sum_{k=1}^p \mathbf{E} |a_{lk}^{(\mathbb{J}, j, \nu+1)}|^q. \end{aligned} \quad (6.21)$$

Using the definition (6.5) of $Q_{\nu 1}^{(\mathbb{J}, j)}$ and the definition (6.6) of coefficients $a_{ll}^{(\mathbb{J}, j, \nu+1)}$, it is straightforward to check that

$$\mathbf{E}|Q_{\nu 1}^{(\mathbb{J}, l)}|^q \leq \mathbf{E}\left[\sum_{k, l=1}^p |a_{kl}^{(\mathbb{J}, j, \nu)}|^2\right]^q. \quad (6.22)$$

Combining (6.18), (6.21) and (6.22), we get by (6.17)

$$\begin{aligned} \mathbf{E}|Q_{\nu}^{(\mathbb{J}, j)}|^q &\leq C_2^q q^q \mathbf{E}|Q_{\nu+1}^{(\mathbb{J}, q)}|^{\frac{q}{2}} + C_1^q C_2^q q^{2q} n^{\frac{q}{4}} \frac{1}{n} \sum_{l=1}^p \mathbf{E}\left(\sum_{k=1}^p |a_{kl}^{(\mathbb{J}, j, \nu+1)}|^2\right)^{\frac{q}{2}} \\ &\quad + C_1^q C_2^q q^{2q} n^{\frac{q}{2}} \frac{1}{n^2} \sum_{l, k=1}^p \mathbf{E}|a_{kl}^{(\mathbb{J}, j, \nu+1)}|^q \\ &\quad + C_1^q C_2^q \mathbf{E}\left[\sum_{q, r \in \mathbb{T}_{\mathbb{J}, k}} |a_{qr}^{(\mathbb{J}, j, \nu)}|^2\right]^q \\ &\quad + C_1^p C_2^q q^q \left(\mathbf{E}\left(\sum_{l=1}^p |a_{ll}^{(\mathbb{J}, j, \nu+1)}|^2\right)^{\frac{q}{2}} + n^{\frac{q}{2}} \frac{1}{n} \sum_{l=1}^p \mathbf{E}|a_{ll}^{(\mathbb{J}, j, \nu+1)}|^q\right). \end{aligned}$$

Applying now Lemma 6.2 and Corollary 6.5, we obtain

$$\begin{aligned} \mathbf{E}|Q_{\nu}^{(\mathbb{J}, j)}|^q &\leq C_3^q q^q \mathbf{E}|Q_{\nu+1}^{(\mathbb{J}, k)}|^{\frac{q}{2}} + C_3^q \mathbf{E}(\Psi^{(\mathbb{J})})^{(2^{\nu}-1)q} v^{-(2^{\nu}-1)q} \\ &\quad + C_3^q q^{2q} v^{-2^{\nu-1}q} n^{-\frac{q}{4}} \mathbf{E}(\Psi^{(\mathbb{J})})^{(2^{\nu}-1)\frac{q}{2}} \left(\frac{1}{n} \sum_{l=1}^p |R_{l+n, l+n}^{(\mathbb{J}, j)}|^{\frac{q}{2}}\right) \\ &\quad + C_3^q q^{2q} n^{-\frac{q}{2}} v^{-2^{\nu}q} \mathbf{E}(\Psi^{(\mathbb{J})})^{(2^{\nu}-1)q} \left(\frac{1}{n} \sum_{l=1}^p |R_{ll}^{(\mathbb{J}, j)}|^{\frac{q}{2}}\right)^2, \quad (6.23) \end{aligned}$$

where $C_3 = 3C_1C_2$. Applying Cauchy–Schwartz and Jensen inequalities, we may rewrite the last inequality in the form

$$\begin{aligned} \mathbf{E}|Q_{\nu}^{(\mathbb{J}, j)}|^q &\leq C_3^q q^q \mathbf{E}|Q_{\nu+1}^{(\mathbb{J}, j)}|^{\frac{q}{2}} + C_3^q \mathbf{E}(\Psi^{(\mathbb{J})})^{(2^{\nu}-1)q} v^{-(2^{\nu}-1)q} \\ &\quad + C_3^q q^{2q} v^{-2^{\nu-1}q} n^{-\frac{q}{4}} \mathbf{E}^{\frac{1}{2}}(\Psi^{(\mathbb{J})})^{(2^{\nu}-1)q} \mathbf{E}^{\frac{1}{2}}\left(\frac{1}{n} \sum_{l=1}^p |R_{l+n, l+n}^{(\mathbb{J}, j)}|^q\right) \\ &\quad + C_3^q q^{2q} n^{-\frac{q}{2}} v^{-2^{\nu}q} \mathbf{E}^{\frac{1}{2}}(\Psi^{(\mathbb{J})})^{(2^{\nu}-1)2q} \mathbf{E}^{\frac{1}{2}}\left(\frac{1}{n} \sum_{l=1}^p |R_{l+n, l+n}^{(\mathbb{J}, j)}|^{2q}\right). \quad (6.24) \end{aligned}$$

Introduce the notation

$$\Gamma_q(z) := \mathbf{E}^{\frac{1}{2}}\left(\frac{1}{n} \sum_{l=1}^p |R_{l+n, l+n}^{(\mathbb{J}, k)}|^{2q}\right).$$

We rewrite the inequality (6.24) using $\Gamma_q(z)$ and the notations of (6.13) as follows

$$\begin{aligned} T_{\nu,q}^{(\mathbb{J},j)} &\leq (C_3q)^q T_{\nu+1,q/2}^{(\mathbb{J},j)} + C_3^q A_{\nu,q}^{(\mathbb{J})} v^{-(2^\nu-1)q} \\ &\quad + (C_3q^2)^q \left(v^{-2^\nu q} n^{-\frac{q}{4}} (A_{\nu,\frac{q}{2}}^{(\mathbb{J})})^{\frac{1}{2}} \Gamma_{\frac{q}{2}}^{\frac{1}{2}}(z) + v^{-2^\nu q} n^{-\frac{q}{2}} A_{\nu,q}^{(\mathbb{J})} \Gamma_q(z) \right). \end{aligned} \quad (6.25)$$

Note that

$$A_{0,q}^{(\mathbb{J})} = 1, \quad A_{\nu,q/2^\nu}^{(\mathbb{J})} \leq \sqrt{1 + \mathbf{E}(\Psi^{(\mathbb{J})})^{2q}} \leq 1 + \mathbf{E}^{\frac{1}{4}}(\Psi^{(\mathbb{J})})^{4q},$$

where $\Psi^{(\mathbb{J})} = \text{Im}(m_n^{(\mathbb{J})}(z) - \frac{p-n+|\mathbb{J}|}{nz}) + \frac{1}{nv}$. Furthermore,

$$\Gamma_{2q/2^\nu} \leq \Gamma_{2q}^{\frac{1}{2^\nu}}.$$

Without loss of generality we may assume $q = 2^L$ and $\nu = 0, \dots, L$. We may write

$$T_{0,q}^{(\mathbb{J},j)} \leq (C_3q)^q T_{1,q/2}^{(\mathbb{J},k)} + C_3^q + (C_3q^2)^q v^{-q} \left(n^{-\frac{q}{4}} \Gamma_{\frac{q}{2}}^{\frac{1}{2}}(z) + n^{-\frac{q}{2}} \Gamma_q(z) \right).$$

By induction we get

$$\begin{aligned} T_{0,q}^{(\mathbb{J},j)} &\leq \prod_{\nu=0}^L (C_3q/2^\nu)^{q/2^\nu} T_{L,1}^{(\mathbb{J},j)} + A_q^{(\mathbb{J})} \sum_{l=1}^L \prod_{\nu=0}^{l-1} (C_3q/2^\nu)^{q/2^\nu} v^{-(2^l-1)q/2^l} \\ &\quad + A_q^{(\mathbb{J})} v^{-q} \sum_{l=1}^L \left(\prod_{\nu=0}^{l-1} (C_3q/2^\nu)^{q/2^\nu} \right) (n^{-q} \Gamma_q^2)^{\frac{1}{2^{l+1}}} \\ &\quad + A_q^{(\mathbb{J})} \sum_{l=1}^L \left(\prod_{\nu=0}^{l-1} (C_3q/2^\nu)^{q/2^\nu} \right) (n^{-q} \Gamma_q^2)^{\frac{1}{2^l}}. \end{aligned} \quad (6.26)$$

It is straightforward to check that

$$\sum_{\nu=1}^{l-1} \frac{\nu}{2^\nu} = 2 \left(1 - \frac{2l+1}{2^l} \right).$$

Note that, for $l \geq 1$,

$$\prod_{\nu=0}^{l-1} (C_3(q/2^\nu))^{q/2^\nu} = \frac{(C_3q)^{2q(1-2^{-l})}}{2^{2q(1-\frac{2l+1}{2^l})}} = 2^{4q\frac{l}{2^l}} \left(\frac{C_3q}{2} \right)^{2q(1-2^{-l})}. \quad (6.27)$$

Applying this relation, we get

$$A_q^{(\mathbb{J})} \sum_{l=0}^L \left(\prod_{\nu=0}^{l-1} (C_3 q / 2^\nu)^{q/2^\nu} \right) v^{-(2^l-1)q/2^l} \leq A_q^{(\mathbb{J})} \left(\frac{C_3 q}{2} \right)^{2q} v^{-q} \sum_{l=0}^{L-1} 2^{\frac{4ql}{2^l}} \left(\frac{4v}{C_3^2 q^2} \right)^{\frac{q}{2^l}}.$$

Note that for $l \geq 0$, $\frac{l}{2^l} \leq \frac{1}{2}$ and recall that $q = 2^L$. Using this observation, we get

$$A_q^{(\mathbb{J})} \sum_{l=0}^L \left(\prod_{\nu=0}^{l-1} (C_3 q / 2^\nu)^{q/2^\nu} \right) v^{-(2^l-1)q/2^l} \leq A_q^{(\mathbb{J})} \left(\frac{C_3 q}{2} \right)^{2q} v^{-q} 2^{2q} \sum_{l=0}^{L-1} \left(\frac{4v}{C_3^2 q^2} \right)^{2^{L-l}}.$$

This implies that for $\frac{4v}{C_3^2 q^2} \leq \frac{1}{2}$,

$$A_q^{(\mathbb{J})} \sum_{l=1}^L \left(\prod_{\nu=0}^{l-1} (C_3 q / 2^\nu)^{q/2^\nu} \right) v^{-(2^l-1)q/2^l} \leq (C_3 q)^{2q} A_q^{(\mathbb{J})} v^{-q}.$$

Furthermore, by definition of $T_{\nu,q}$, we have

$$T_{L,1}^{(\mathbb{J},j)} = \mathbf{E} Q_L^{(\mathbb{J},j)} \leq \mathbf{E} \sum_{l,k=1}^p (a_{kl}^{(\mathbb{J},j,L)})^2.$$

Applying Corollary 6.3 and Hölder's inequality, we get

$$T_{L,1}^{(\mathbb{J},j)} \leq E(v^{-1} \Psi^{(\mathbb{J})})^q \leq v^{-q} A_q^{(\mathbb{J})}. \quad (6.28)$$

By condition (6.15), we have

$$\Gamma_q := \Gamma_q(u + iv) \leq s_0^{2q} C_0^{2q}.$$

Using this inequality, we get,

$$\begin{aligned} A_q^{(\mathbb{J})} v^{-q} \sum_{l=1}^L \left(\prod_{\nu=0}^{l-1} (C_3 q / 2^\nu)^{q/2^\nu} \right) n^{-\frac{q}{2^{l+1}}} \Gamma_q^{\frac{1}{2^l}} \\ \leq A_q^{(\mathbb{J})} v^{-q} \sum_{l=0}^L \left(\prod_{\nu=0}^{l-1} (C_3 q / 2^\nu)^{q/2^\nu} \right) (s_0^4 C_0^4 n^{-1})^{\frac{q}{2^{l+1}}}. \end{aligned} \quad (6.29)$$

Applying relation (6.27), we obtain

$$\begin{aligned}
 A_q^{(\mathbb{J})} v^{-q} \sum_{l=1}^L \left(\prod_{\nu=0}^{l-1} (C_3 q / 2^\nu)^{q/2^\nu} \right) n^{-\frac{q}{2^{l+2}}} \Gamma_q^{\frac{1}{2^{l+2}}} \\
 \leq \left(\frac{C_3 q}{2} \right)^{2q} A_q^{(\mathbb{J})} v^{-q} \sum_{l=1}^L 2^{2q \frac{l}{2^l}} \left(\frac{C_3 q}{2} \right)^{-\frac{2q}{2^l}} (s_0^2 C_0^2 n^{-1})^{\frac{q}{2^{l+2}}} \\
 = \left(\frac{C_3 q}{2} \right)^{2q} A_q^{(\mathbb{J})} v^{-q} \sum_{l=1}^L 2^{q \frac{l}{2^{l-1}}} \left(\frac{C_3 q}{2} \right)^{-\frac{2q}{2^{l-1}}} ((s_0^4 C_0^4 n^{-1})^{\frac{1}{4}})^{\frac{q}{2^{l-1}}} \\
 = \left(\frac{C_3 q}{\sqrt{2}} \right)^{2q} A_q^{(\mathbb{J})} v^{-q} \sum_{l=1}^L \left(\frac{s_0 C_0}{C_3 q n^{\frac{1}{4}}} \right)^{2^{L-l+1}}.
 \end{aligned}$$

Without loss of generality we may assume that $C_3 \geq 2(C_0 s_0)$. Then we get

$$A_q^{(\mathbb{J})} v^{-q} \sum_{l=0}^L \left(\prod_{\nu=0}^{l-1} (C_3 q / 2^\nu)^{q/2^\nu} \right) n^{-\frac{q}{2^{l+1}}} \Gamma_q^{\frac{1}{2^l}} \leq (C_3 q)^{2q} A_q^{(\mathbb{J})} v^{-q}.$$

Analogously we get

$$A_q^{(\mathbb{J})} v^{-q} \sum_{l=1}^L \left(\prod_{\nu=0}^{l-1} (C_3 p / 2^\nu)^{q/2^\nu} \right) n^{-\frac{q}{2^l}} \Gamma_q^{\frac{1}{2^{l-1}}} \leq (C_3 q)^{2q} v^{-q} A_q^{(\mathbb{J})}. \quad (6.30)$$

Combining inequalities (6.26), (6.28), (6.29), (6.30), we finally arrive at

$$T_{0,q}^{(\mathbb{J},j)} \leq 6(C_3 q)^{2q} v^{-q} A_q^{(\mathbb{J})}. \quad (6.31)$$

Thus, Lemma 6.6 is proved. □

□

6.2 Diagonal Entries of the Resolvent Matrix

Recall that

$$R_{jj} = -\frac{1}{z + y m_n(z) + \frac{y-1}{z}} + \frac{1}{z + y m_n(z) \frac{y-1}{z}} \varepsilon_j R_{jj}, \quad (6.32)$$

or

$$R_{jj} = -\frac{1}{z + y s_y(z) + \frac{y-1}{z}} + \frac{y \Lambda_n R_{jj}}{(z + y s_y(z) + \frac{y-1}{z})} + \frac{1}{z + y s_y(z) + \frac{y-1}{z}} \varepsilon_j R_{jj}, \quad (6.33)$$

where $\varepsilon_j := \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3}$ with

$$\begin{aligned}\varepsilon_{j1} &:= -\frac{1}{p} \sum_{k \neq l \in \mathbb{T}_j} X_{jk} X_{jl} R_{kl}^{(j)}, \quad \varepsilon_{j2} := -\frac{1}{p} \sum_{k \in \mathbb{T}_j} (X_{jk}^2 - 1) R_{kk}^{(j)}, \\ \varepsilon_{j3} &:= \frac{1}{p} \left(\sum_{l=1}^p R_{l+n, l+n}^{(j)} - \sum_{l=1}^p R_{l+n, l+n} \right), \\ \Lambda_n &:= m_n(z) - s_y(z) = \frac{1}{2n} \text{Tr } \mathbf{R} - s_y(z) + \frac{1-y}{2yz}.\end{aligned}\tag{6.34}$$

Using equation (1.10), we may rewrite representation (6.33) as follows

$$R_{jj} = s_y(z) + y \Lambda_n R_{jj} s_y(z) + s_y(z) \varepsilon_j R_{jj}.\tag{6.35}$$

Similar we have, for $j = 1, \dots, p$

$$R_{j+n, j+n} = -\frac{1}{z + y m_n(z)} + \frac{1}{z + y m_n(z)} \widehat{\varepsilon}_j R_{jj},\tag{6.36}$$

where

$$\begin{aligned}\widehat{\varepsilon}_{j1} &:= -\frac{1}{p} \sum_{k=1}^n (X_{kj}^2 - 1) R_{kk}^{(j+n)}, \quad \widehat{\varepsilon}_{j2} := -\frac{1}{p} \sum_{1 \leq k \neq l \leq n} X_{kj} X_{lj} R_{kl}^{(j+n)}, \\ \widehat{\varepsilon}_{j3} &:= \frac{1}{p} \left(\sum_{l=1}^n R_{l, l}^{(j+n)} - \sum_{l=1}^n R_{l, l} \right).\end{aligned}\tag{6.37}$$

We may rewrite (6.36) as

$$R_{j+n, j+n} = -\frac{1}{z + y s_y(z)} + \frac{y \Lambda_n R_{j+n, j+n}}{z + y s_y(z)} + \frac{1}{z + y s_y(z)} \widehat{\varepsilon}_j R_{j+n, j+n}.$$

where

$$\Lambda_n := m_n(z) - s_y(z) = \frac{1}{2n} \text{Tr } \mathbf{R} - s_y(z) + \frac{1-y}{2yz}.\tag{6.38}$$

We shall consider the representation (6.35) only, because arguments for the representation (6.36) is similar. In the latter case (6.36) we need to use Lemma 8.9 instead of the bound $|s_y(z)| \leq \frac{1}{\sqrt{y}}$. Since $|s_y(z)| \leq 1/\sqrt{y}$, the representation (6.35) yields, for any $q \geq 1$,

$$|R_{jj}^{(\mathbb{J})}|^q \leq 3^q y^{-\frac{q}{2}} + 3^q y^{-\frac{q}{2}} |\varepsilon_j^{(\mathbb{J})}|^q |R_{jj}^{(\mathbb{J})}|^q + 3^p y^{-\frac{q}{2}} |\Lambda_n^{(\mathbb{J})}|^q |R_{jj}^{(\mathbb{J})}|^q.\tag{6.39}$$

We shall use the equality

$$\mathrm{Tr} \mathbf{R} - \mathrm{Tr} \mathbf{R}^{(j)} = (1 + \frac{1}{p} \sum_{k,l=1}^p X_{jk} X_{jl} [(\mathbf{R}^{(j)})^2]_{k+n,l+n}) R_{jj}. \quad (6.40)$$

See for instance [12], Lemma 3.1.

Applying the Cauchy – Schwartz inequality, we get

$$3^{-q} y^{\frac{q}{2}} \mathbf{E} |R_{jj}^{(\mathbb{J})}|^q \leq 1 + \mathbf{E}^{\frac{1}{2}} |\varepsilon_j^{(\mathbb{J})}|^{2q} \mathbf{E}^{\frac{1}{2}} |R_{jj}^{(\mathbb{J})}|^{2q} + \mathbf{E}^{\frac{1}{2}} |\Lambda_n^{(\mathbb{J})}|^{2q} \mathbf{E}^{\frac{1}{2}} |R_{jj}^{(\mathbb{J})}|^{2q}. \quad (6.41)$$

We shall investigate now the behavior of $\mathbf{E} |\varepsilon_j^{(\mathbb{J})}|^{2q}$ and $\mathbf{E} |\Lambda_n^{(\mathbb{J})}|^{2q}$. First we note,

$$\mathbf{E} |\varepsilon_j^{(\mathbb{J})}|^{2q} \leq 3^{2q} \sum_{\nu=1}^3 \mathbf{E} |\varepsilon_{j\nu}^{(\mathbb{J})}|^{2q}.$$

Lemma 6.7. *Assuming the conditions of Theorem 1.3 we have, for any $q \geq 1$, and for any $z = u + iv \in \mathbb{C}_+$,*

$$\mathbf{E} |\varepsilon_{j3}^{(\mathbb{J})}|^{2q} \leq \frac{1}{n^{2q} v^{2q}}.$$

Proof. For a proof of this Lemma see Lemma 3.2 in [10]. \square

Let $A_1 > 0$ and $0 \leq v_1 \leq 4\sqrt{y}$ be a fixed.

Lemma 6.8. *Assuming the conditions of Theorem 1.1, and assuming for all $\mathbb{J} \subset T$ with $|\mathbb{J}| \leq L$ and all $l \in \{n+1, \dots, n+p\}$*

$$\mathbf{E} |R_{ll}^{(\mathbb{J})}|^q \leq C_0^q, \text{ for } 1 \leq q \leq A_1 (nv)^{\frac{1}{4}} \text{ and for } v \geq v_1, \quad (6.42)$$

and for all $\widehat{\mathbb{J}} \subset \widehat{\mathbb{T}}$ with $|\widehat{\mathbb{J}}| \leq L$ and all $l \in \{n+1, \dots, n+p\}$

$$\mathbf{E} |R_{ll}^{(\widehat{\mathbb{J}})}|^q \leq C_0^q, \text{ for } 1 \leq q \leq A_1 (nv)^{\frac{1}{4}} \text{ and for } v \geq v_1, \quad (6.43)$$

we have, for all $v \geq v_1/s_0$, and for all $\mathbb{J} \subset \mathbb{T}$ with $|\mathbb{J}| \leq L-1$,

$$\max\{\mathbf{E} |\varepsilon_{j1}^{(\mathbb{J})}|^{2q}, \mathbf{E} |\widehat{\varepsilon}_{j1}^{(\mathbb{J})}|^{2q}\} \leq (C_1 q)^{2q} n^{-q} s_0^{2q} C_0^{4q}, \text{ for } 1 \leq q \leq A_1 (nv)^{\frac{1}{4}}.$$

Proof. Recall that $s_0 = 2^4$ and note that if $q \leq A_1 (nv)^{\frac{1}{4}}$ for $v \geq v_1/s_0$ then $q' = 2q \leq A_1 (nv)^{\frac{1}{4}}$ for $v \geq v_1$. Let $v' := vs_0$. If $v \geq v_1/s_0$ then $v' \geq v_1$. We have

$$q' = 2q \leq 2A_1 (nv)^{\frac{1}{4}} = 2A_1 (nv' s_0^{-1})^{\frac{1}{4}} = A_1 (nv')^{\frac{1}{4}}. \quad (6.44)$$

We apply now Rosenthal's inequality for the moments of sums of independent random variables and get

$$\mathbf{E}|\varepsilon_{j1}^{(\mathbb{J})}|^{2q} \leq (C_1 q)^{2q} n^{-2q} \left(\mathbf{E} \left(\sum_{l=1}^p |R_{l+n, l+n}^{(\mathbb{J}, j)}|^2 \right)^q + \mathbf{E}|X_{jl}|^{4q} \sum_{l=1}^p \mathbf{E}|R_{l+n, l+n}^{(\mathbb{J}, j)}|^{2q} \right).$$

According to inequality (6.44) we may apply Lemma 8.5 and condition (6.42) for $q' = 2q$. We get, for $v \geq v_1/s_0$,

$$\mathbf{E}|\varepsilon_{j1}^{(\mathbb{J})}|^{2q} \leq (C_1 q)^{2q} n^{-q} s_0^{2q} C_0^{2q}.$$

We use as well that by the conditions of Theorem 1.1, $\mathbf{E}|X_{jl}|^{4q} \leq D^{4q-4} n^{q-1} \mu_4$, and by Jensen's inequality, $(\frac{1}{n} \sum_{l=1}^p |R_{l+n, l+n}^{(\mathbb{J}, j)}|^2)^q \leq \frac{1}{n} \sum_{l=1}^p |R_{l+n, l+n}^{(\mathbb{J}, j)}|^{2q}$. Similar we get the estimation for $\mathbf{E}|\widehat{\varepsilon}_{j1}^{(\mathbb{J})}|^{2q}$. Thus, Lemma 6.8 is proved. \square

Lemma 6.9. *Assuming the conditions of Theorem 1.1, condition (6.42), for $v \geq v_1$ and $q \leq A_1(nv)^{\frac{1}{4}}$, we have, for any $v \geq v_1/s_0$ and $q \leq A_1(nv)^{\frac{1}{4}}$,*

$$\max\{\mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2q}, \mathbf{E}|\widehat{\varepsilon}_{j2}^{(\mathbb{J})}|^{2q}\} \leq 6(C_3 q)^{4q} n^{-q} v^{-q} A_q^{(\mathbb{J})} + 2(C_3 q)^{4q} n^{-q} v^{-q} (C_0 s_0)^q.$$

Proof. We consider the quantity $\mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2q}$ only. The other one is similar. We apply Burkholder's inequality for quadratic forms. See Lemma 8.3 in the Appendix. We obtain

$$\begin{aligned} \mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2q} &\leq (C_1 q)^{2q} n^{-2q} \left(\mathbf{E} \left(\sum_{l=1}^p \left| \sum_{k=1}^{l-1} X_{jk} R_{l+n, k+n}^{(\mathbb{J}, j)} \right|^2 \right)^q \right. \\ &\quad \left. + \max_{j, l} \mathbf{E}|X_{jl}|^{2q} \sum_{l=1}^p \mathbf{E} \left| \sum_{k=1}^{l-1} X_{jk} R_{l+n, k+n}^{(\mathbb{J}, j)} \right|^{2q} \right). \end{aligned}$$

Using now the quantity $Q_0^{(\mathbb{J}, j)}$ for the first term and Rosenthal's inequality and condition (1.2) for the second term, we obtain with Lemma 8.4,

inequality (8.8), in the Appendix and $C_3 = C_1 C_2$

$$\begin{aligned}
\mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2q} &\leq (C_2 q)^{2q} n^{-q} \mathbf{E}|Q_0^{(\mathbb{J},j)}|^q \\
&\quad + (C_3 q)^{4q} n^{-\frac{3q}{2}} \frac{1}{n} \sum_{l=1}^p \mathbf{E} \left(\sum_{k=1}^{l-1} |R_{l+n,k+n}^{(\mathbb{J},j)}|^2 \right)^q \\
&\quad + (C_3 q)^{4q} n^{-q} \frac{1}{n^2} \sum_{l=1}^p \sum_{k=1}^{l-1} \mathbf{E} |R_{l+n,k+n}^{(\mathbb{J},j)}|^{2q} \\
&\leq (C_2 q)^{2q} n^{-q} \mathbf{E}|Q_0^{(\mathbb{J},j)}|^q + (C_3 q)^{4q} n^{-\frac{3q}{2}} v^{-q} \frac{1}{n} \sum_{l=1}^p \mathbf{E} |R_{l+n,l+n}^{(\mathbb{J},j)}|^q \\
&\quad + (C_3 q)^{4q} n^{-q} v^{-q} \frac{1}{n^2} \sum_{l=1}^p \mathbf{E} |R_{l+n,l+n}^{(\mathbb{J},j)}|^q.
\end{aligned}$$

By Lemma 8.5 and condition (6.42), we get

$$\mathbf{E}|\varepsilon_{j2}^{(\mathbb{J})}|^{2q} \leq (C_3 q)^{2q} n^{-q} \mathbf{E}|Q_0^{(\mathbb{J},j)}|^q + 2(C_3 q)^{3q} n^{-q} v^{-q} (C_0 s_0)^q.$$

Applying now Lemma 6.6, we get the claim. Thus, Lemma 6.9 is proved. \square

Recall that

$$\Lambda_n^{(\mathbb{J})} = \frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} R_{jj}^{(\mathbb{J})} - s_y(z), \quad \text{and} \quad T_n^{(\mathbb{J})}(z) = \frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \varepsilon_j^{(\mathbb{J})} R_{jj}^{(\mathbb{J})}.$$

and

$$\Lambda_n^{(\widehat{\mathbb{J}})} = \frac{1}{n} \sum_{j=1}^n R_{jj}^{(\mathbb{J})} - s_y(z) + \frac{(p - |J| - n)_+}{nz}, \quad \text{and} \quad T_n^{(\mathbb{J})}(z) = \frac{1}{n} \sum_{j=1}^n \varepsilon_j^{(\widehat{\mathbb{J}})} R_{jj}^{(\mathbb{J})}.$$

Lemma 6.10. *Assuming the conditions of Theorem 1.1, we have*

$$|\Lambda_n^{(\mathbb{J})}| \leq C(\sqrt{|T_n^{(\mathbb{J})}(z)|} + \frac{\sqrt{|\mathbb{J}|}}{\sqrt{n}}).$$

Proof. It is similar to the inequality (2.10) in [20]. For completeness we include short proof here. Obviously

$$\Lambda_n^{(\mathbb{J})}(z) = \frac{(T_n^{(\mathbb{J})}(z) - \frac{|\mathbb{J}|}{n})}{z + 2ys_y(z) + \frac{y-1}{z} - y\Lambda_n^{(\mathbb{J})}(z)}. \quad (6.45)$$

Note that

$$z + 2ys_y(z) + \frac{y-1}{z} = \sqrt{\left(z + \frac{y-1}{z}\right)^2 - 4y}.$$

Solving this equation (6.45) w.r.t. $\Lambda_n^{(\mathbb{J})}(z)$, we get

$$\Lambda_n^{(\mathbb{J})}(z) = \frac{-\sqrt{\left(z + \frac{y-1}{z}\right)^2 - 4y} + \sqrt{\left(z + \frac{y-1}{z}\right)^2 - 4y - 4y\tilde{T}_n^{(\mathbb{J})}(z)}}{2}, \quad (6.46)$$

where

$$\tilde{T}_n(z) = s_y(z)(T_n^{(\mathbb{J})}(z) - \frac{|\mathbb{J}|}{n}).$$

We take here the branch of \sqrt{z} such that $\text{Im} \sqrt{z} \geq 0$. Since for any $a, b \in \mathbb{C}$ $|\sqrt{a+b} - \sqrt{a}| \leq C \frac{|b|}{\sqrt{|a|+|b|}}$, we get

$$|\Lambda_n^{(\mathbb{J})}(z)| \leq C(\sqrt{|T_n^{(\mathbb{J})}(z)|} + \frac{\sqrt{|\mathbb{J}|}}{\sqrt{n}}).$$

Thus, Lemma 6.10 is proved. \square

Lemma 6.11. *Assuming the conditions of Theorem 1.1 and condition (6.42), we obtain, for $|\mathbb{J}| \leq Cn^{\frac{1}{2}}$*

$$\begin{aligned} \mathbf{E}|\Lambda_n^{(\mathbb{J})}|^{2q} &\leq \frac{C^q}{n^{\frac{q}{4}}} + \left(\frac{\mu_4}{n^{\frac{q}{2}+1}} + \frac{1}{n^{2q}v^{2q}} + (C_1q)^{2q}n^{-q}s_0^{2q}C_0^{4q} \right. \\ &\quad \left. + 6(C_3q)^{4q}n^{-q}v^{-q}A_q^{(\mathbb{J})} + 2(C_3q)^{4q}n^{-q}v^{-q}(C_0s_0)^q \right)^{\frac{1}{2}} (C_0s_0)^q. \end{aligned}$$

Proof. By Lemma 6.9, we have

$$\mathbf{E}|\Lambda_n^{(\mathbb{J})}|^{2q} \leq C^p \mathbf{E}|T_n^{(\mathbb{J})}(z)|^q + \frac{|\mathbb{J}|^{\frac{q}{2}}}{n^{\frac{q}{2}}} \leq C^q \mathbf{E}|T_n^{(\mathbb{J})}(z)|^q + \frac{C}{n^{\frac{q}{4}}}.$$

Furthermore,

$$\mathbf{E}|T_n^{(\mathbb{J})}(z)|^q \leq \left(\frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \mathbf{E}|\varepsilon_j^{(\mathbb{J})}|^{2q} \right)^{\frac{1}{2}} \left(\frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \mathbf{E}|R_{jj}^{(\mathbb{J})}|^{2q} \right)^{\frac{1}{2}}.$$

Lemmas 6.7 – 6.9 together with Lemma 8.5 imply

$$\begin{aligned} \frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \mathbf{E}|\varepsilon_j^{(\mathbb{J})}|^{2q} &\leq 4^{2q-1} \left(\frac{\mu_4}{n^{\frac{q}{2}+1}} + \frac{1}{n^{2q}v^{2q}} + (C_1q)^{2q}n^{-q}s_0^{2q}C_0^{4q} \right. \\ &\quad \left. + 6(C_3q)^{4q}n^{-q}v^{-q}A_q^{(\mathbb{J})} + 2(C_3q)^{4q}n^{-q}v^{-q}(C_0s_0)^q \right). \end{aligned} \quad (6.47)$$

Thus, Lemma 6.11 is proved. \square

Lemma 6.12. *Assuming the conditions of Theorem 1.1 and condition (6.42), there exists an absolute constant C_4 such that, for $q \leq A_1(nv)^{\frac{1}{4}}$ and $v \geq v_1/s_0$, we have, uniformly in $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq Cn^{\frac{1}{2}}$, and for $z \in \mathbb{G}$,*

$$A_q^{(\mathbb{J})} \leq C_4^q.$$

Proof. We start from the obvious inequality, using $|s_y(z)| \leq 1/\sqrt{y}$,

$$\mathbf{E}|\Psi^{(\mathbb{J})}|^{2q} \leq 3^{2q} \left(\frac{1}{y^q} + \left(\frac{v|\mathbb{J}|}{n|z|^2} \right)^{2q} + \left(\frac{v(1-y)}{y|z|^2} \right)^{2q} + (nv)^{-2q} + \mathbf{E}|\Lambda_n^{(\mathbb{J})}(z)|^{2q} \right).$$

Note that, for $z \in \mathbb{G}$,

$$\frac{v|\mathbb{J}|}{n|z|^2} \leq \frac{|\mathbb{J}|}{n|z|} \leq \frac{C}{\sqrt{n}|z|} \leq \frac{C}{\sqrt{nv}} \leq \frac{1}{2\sqrt{y}}. \quad (6.48)$$

It is straightforward to check that, for $z \in \mathbb{G}$,

$$\frac{v(1-y)}{y|z|^2} \leq \frac{1-y}{y|z|} \leq \frac{1+\sqrt{y}}{y}. \quad (6.49)$$

Furthermore, applying Lemma 6.11, we get

$$\begin{aligned} \mathbf{E}|\Psi^{(\mathbb{J})}|^{2q} &\leq 3^{2q} \left(\left(\frac{2}{y} \right)^{2q} + (nv)^{-2q} + \left(\frac{\mu_4}{n^{\frac{q}{2}+1}} + \frac{1}{n^{2q}v^{2q}} + (C_1q)^{2q}n^{-q}s_0^{2q}C_0^{4q} \right. \right. \\ &\quad \left. \left. + 6(C_3q)^{4q}n^{-q}v^{-q}A_q^{(\mathbb{J})} + 2(C_3q)^{4q}n^{-q}v^{-q}(C_0s_0)^q \right)^{\frac{1}{2}} (C_0s_0)^q \right). \end{aligned} \quad (6.50)$$

By definition,

$$A_q^{(\mathbb{J})} \leq 1 + \mathbf{E}^{\frac{1}{2}}(\Psi^{(\mathbb{J})})^{2q}. \quad (6.51)$$

Inequalities (6.51) and (6.50) together imply

$$\begin{aligned} A_q^{(\mathbb{J})} &\leq 1 + 3^q \left(1 + (nv)^{-\frac{q}{2}} + (C_0s_0)^{\frac{q}{2}} \left(\mu_4^{\frac{1}{4}}n^{-\frac{q}{8}} + \frac{1}{n^{\frac{q}{2}}v^{\frac{q}{2}}} + (C_1q)^{\frac{q}{2}}n^{-\frac{q}{4}}s_0^{\frac{q}{2}}C_0^q \right. \right. \\ &\quad \left. \left. + 3(C_3q)^qn^{-\frac{q}{4}}v^{-\frac{q}{4}}(A_q^{(\mathbb{J})})^{\frac{1}{4}} + 2(C_3q)^qn^{-\frac{q}{4}}v^{-\frac{q}{4}}(C_0s_0)^{\frac{q}{4}} \right) \right). \end{aligned}$$

Let $C' = s_0 \max\{9, C_3^{\frac{1}{2}}, C_0^{\frac{3}{2}} C_1^{\frac{1}{2}}, 3C_3 C_0^{\frac{1}{2}}, C_3 C_0^{\frac{3}{4}}\}$. The last inequality implies that

$$A_q^{(\mathbb{J})} \leq C'^q \left(1 + (nv)^{-\frac{q}{2}} + \mu_4^{\frac{1}{2}} n^{-\frac{q}{8}} + \frac{1}{n^{\frac{q}{2}} v^{\frac{q}{2}}} + q^{\frac{q}{2}} n^{-\frac{q}{4}} + q^q n^{-\frac{q}{4}} v^{-\frac{q}{4}} + q^{\frac{4q}{3}} n^{-\frac{q}{3}} v^{-\frac{q}{3}} \right).$$

For $q \leq A_1(nv)^{\frac{1}{4}}$, we get, for $z \in \mathbb{G}$,

$$A_q^{(\mathbb{J})} \leq C_4^q,$$

where C_4 is some absolute constant. We may take $C_4 = 2C'$. \square

Corollary 6.13. *Assuming the conditions of Theorem 1.1 and condition (6.42), we have, for $v \geq v_1/s_0$, and for any $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq \sqrt{n}$*

$$\mathbf{E}|\Lambda_n^{(\mathbb{J})}|^{2q} \leq C_0^{2q} \left(\frac{4^{\frac{q}{4}} \mu_4^{\frac{1}{2}} s_0^q}{n^{\frac{q}{4}} v^{\frac{q}{4}}} + \frac{s_0^{\frac{q}{2}}}{n^q v^q} + \frac{C_5^q q^{2q}}{n^{\frac{q}{2}} v^{\frac{q}{2}}} \right), \quad (6.52)$$

where

$$C_5 := 4C_1^2 s_0^4 + 6^{\frac{1}{q}} C_3^4 C_4 + 2^{\frac{1}{q}} C_3^4 s_0^3.$$

Proof. Without loss of generality we may assume that $C_0 > 1$. The bound (6.52) follows now from Lemmas 6.11 and 6.12. \square

Lemma 6.14. *Assuming the conditions of Theorem 1.1 and condition (6.42) for $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq L \leq \sqrt{n}$, there exist positive constant A_0, C_0, A_1 depending on μ_4, D only, such that we have, for $q \leq A_1(nv)^{\frac{1}{4}}$ and $v \geq v_1/s_0$ uniformly in \mathbb{J} and v_1 , for $j \in \mathbb{T}_{\mathbb{J}} \cup \{n+1, \dots, n+p\}$*

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^q \leq C_0^q$$

and for $\widehat{\mathbb{J}} \subset \widehat{\mathbb{T}}$ such that $|\widehat{\mathbb{J}}| \leq L \leq \sqrt{n}$, there exist positive constant A_0, C_0, A_1 depending on μ_4, D only, such that we have, for $q \leq A_1(nv)^{\frac{1}{4}}$ and $v \geq v_1/s_0$ uniformly in $\widehat{\mathbb{J}}$ and v_1 , for any $j \in \{1, \dots, n\} \cup \widehat{\mathbb{T}}_{\widehat{\mathbb{J}}}$,

$$\mathbf{E}|R_{jj}^{(\widehat{\mathbb{J}})}|^q \leq C_0^q$$

with $|\widehat{\mathbb{J}}| \leq L - 1$.

Proof. According to inequality (6.41), we have

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^q \leq 4^q(1 + (\mathbf{E}^{\frac{1}{2}}|\Lambda_n^{(\mathbb{J})}|^{2q} + \mathbf{E}^{\frac{1}{2}}|\varepsilon_j^{(\mathbb{J})}|^{2q})\mathbf{E}^{\frac{1}{2}}|R_{jj}^{(\mathbb{J})}|^{2q}). \quad (6.53)$$

Applying condition (6.42), we get

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^q \leq 4^q(1 + (\mathbf{E}^{\frac{1}{2}}|\Lambda_n^{(\mathbb{J})}|^{2q} + \mathbf{E}^{\frac{1}{2}}|\varepsilon_{j1}^{(\mathbb{J})}|^{2q} + \dots + \mathbf{E}^{\frac{1}{2}}|\varepsilon_{j4}^{(\mathbb{J})}|^{2q})s_0^q C_0^q).$$

Combining results of Lemmas 6.7 – 6.9 and Corollary 6.13, we obtain

$$\begin{aligned} \mathbf{E}|R_{jj}^{(\mathbb{J})}|^q &\leq 5^q \left(1 + s_0^q C_0^{2q} \left(\frac{4^{\frac{q}{4}} \mu_4^{\frac{1}{4}} s_0^q}{n^{\frac{q}{4}} v^{\frac{q}{4}}} + \frac{s_0^q}{n^q v^q} + \frac{C_5^q q^{2q}}{n^{\frac{q}{2}} v^{\frac{q}{2}}} \right)^{\frac{1}{2}} \right. \\ &\quad \left. + s_0^q C_0^{3q} \left(\frac{4^{\frac{q}{4}} \mu_4^{\frac{1}{4}}}{n^{\frac{q}{4}} v^{\frac{q}{4}}} + \frac{s_0^q}{n^q v^q} + \frac{C_5^q q^{2q}}{n^{\frac{q}{2}} v^{\frac{q}{2}}} \right) \right). \end{aligned} \quad (6.54)$$

We may rewrite the last inequality as follows

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^q \leq C_0^q \left(\frac{5^q}{C_0^q} + \frac{\widehat{C}_1^{\frac{q}{8}}}{(nv)^{\frac{q}{8}}} + \frac{(\widehat{C}_2 q^4)^{\frac{q}{4}}}{(nv)^{\frac{q}{4}}} + \frac{(\widehat{C}_3 q^4)^{\frac{q}{2}}}{(nv)^{\frac{q}{2}}} + \frac{\widehat{C}_4^q}{(nv)^q} \right),$$

where

$$\begin{aligned} \widehat{C}_1 &= 5^8 s_0^{12} C_0^8 \mu_4^{\frac{1}{p}}, \\ \widehat{C}_2 &= 5^4 s_0^4 C_0^4 C_5^2 (1 + 2C_0^4 \mu_4^{\frac{2}{p}}), \\ \widehat{C}_3 &= 5^2 s_0^2 C_0^2 (s_0 + C_5^2), \\ \widehat{C}_4 &= 5C_0^2 s_0^2. \end{aligned}$$

Note that for

$$A_0 \geq 2^8 A_1^4 \max\{\widehat{C}_1, \dots, \widehat{C}_4\} \quad (6.55)$$

and $C_0 \geq 25$, we obtain that

$$\mathbf{E}|R_{jj}^{(\mathbb{J})}|^q \leq C_0^q.$$

Thus Lemma 6.14 is proved. \square

Corollary 6.15. *Assuming the conditions of Theorem 1.1, we have, for $q \leq 8$ and $v \geq v_0 = A_0 n^{-1}$ there exist a constant $C_0 > 0$ depending on μ_4 and D only such that for all $1 \leq j \leq n + p$ and all $z \in \mathbb{G}$*

$$\mathbf{E}|R_{jj}|^q \leq C_0^q, \quad (6.56)$$

and

$$\mathbf{E} \frac{1}{|z + ym_n(z) + \frac{y-1}{z}|^q} \leq C_0^q. \quad (6.57)$$

Proof. Let $L = [-\log_{s_0} v_0] + 1$. Note that $s_0^{-1}v_0 \leq s_0^{-L} \leq v_0$ and $A_1 n^{\frac{1}{4}} s_0^{-\frac{L}{4}} \geq s_0^{-\frac{1}{4}} A_1 (nv_0)^{\frac{1}{4}}$. We may choose $C_0 = 25$ and A_0, A_1 such that (6.55) holds and

$$A_1 (nv)^{\frac{1}{4}} \geq 8. \quad (6.58)$$

Then, for $v = 1$, and for any $q \geq 1$, for any set $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq L$

$$\mathbf{E} |R_{jj}^{(\mathbb{J})}|^q \leq C_0^q. \quad (6.59)$$

By Lemma 6.14, inequality (6.59) holds for $v \geq 1/s_0$ and for $q \leq A_1 n^{\frac{1}{4}}/s_0^{\frac{1}{4}}$ and for $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq L - 1$. After repeated application of Lemma 6.14 (with (6.59) as assumption valid for $v \geq 1/s_0$) we arrive at the conclusion that the inequality (6.59) holds for $v \geq 1/s_0^2$, $p \leq A_1 n^{\frac{1}{4}}/s_0^{\frac{1}{2}}$ and all $\mathbb{J} \subset \mathbb{T}$ such that $|\mathbb{J}| \leq L - 2$. Continuing this iteration inequality, inequality (6.59) finally holds for $v \geq A_0 n^{-1}$, $q \leq 8$ and $\mathbb{J} = \emptyset$.

The proof of inequality of (6.57) is similar. We have by (1.10) ,

$$\begin{aligned} \frac{1}{|z + ym_n(z) + \frac{y-1}{z}|} &\leq \frac{1}{|ys(z) + z + \frac{y-1}{z}|} + \frac{|\Lambda_n|}{|z + ym_n(z) + \frac{y-1}{z}| |z + ys(z) + \frac{y-1}{z}|} \\ &\leq |s(z)| \left(1 + \frac{|\Lambda_n|}{|z + ym_n(z) + \frac{y-1}{z}|}\right). \end{aligned} \quad (6.60)$$

Furthermore, using that $|m'_n(z)| \leq \frac{1}{n} \sum_{j=1}^n \mathbf{E} |R_{jj}|^2$ and $z + ym_n(z) + \frac{y-1}{z} \geq v + y\text{Im} m_n(z) + \frac{v(1-y)}{|z|^2}$, we get

$$\begin{aligned} \left| \frac{d}{dz} \log(z + ym_n(z) + \frac{y-1}{z}) \right| &\leq \frac{|1 + ym'_n(z) - \frac{y-1}{z^2}|}{|z + ym_n(z) + \frac{y-1}{z}|} \\ &\leq \frac{1}{v} \frac{v + y\text{Im} m_n(z) + \frac{(1-y)v}{|z|^2}}{|v + y\text{Im} m_n(z) + \frac{v(1-y)}{|z|^2}|} \leq \frac{1}{v}. \end{aligned}$$

By integration, this implies that (see the proof of Lemma 8.5)

$$\frac{1}{|(u + iv/s_0) + ym_n(u + iv/s_0) + \frac{y-1}{u+iv/s_0}|} \leq \frac{s_0}{|(u + iv) + ym_n(u + iv) + \frac{y-1}{u+iv}|}. \quad (6.61)$$

Inequality (6.60) and the Cauchy–Schwartz inequality together imply

$$\mathbf{E} \frac{1}{|z + ym_n(z) + \frac{y-1}{z}|^q} \leq 2^q |s(z)|^q (1 + \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^{2q} \mathbf{E}^{\frac{1}{2}} \frac{1}{|z + ym_n(z) + \frac{y-1}{z}|^{2q}}).$$

Applying inequality (6.61), we obtain

$$\mathbf{E} \frac{1}{|z + ym_n(z) + \frac{y-1}{z}|^q} \leq 2^q |s(z)|^q (1 + \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^{2q} s_0^q C_0^q).$$

Using Corollary 6.13, we get, for $v \geq 1/s_0$

$$\mathbf{E} \frac{1}{|z + ym_n(z) + \frac{y-1}{z}|^q} \leq 2^q |s(z)|^q \left(1 + \left(\frac{4^{\frac{q}{8}} \mu_4^{\frac{1}{4}} s_0^{\frac{q}{2}}}{n^{\frac{q}{8}} v^{\frac{q}{8}}} + \frac{s_0^{\frac{q}{4}}}{n^{\frac{q}{2}} v^{\frac{q}{2}}} + \frac{C_5^q q^q}{n^{\frac{q}{4}} v^{\frac{q}{4}}} \right) s_0^q C_0^{2q} \right).$$

Thus inequality (6.57) holds for $v \geq 1/s_0$ as well. Repeating this argument inductively with A_0, A_1, C_j satisfying (6.55) for the regions $v \geq s_0^{-\nu}$, for $\nu = 1, \dots, L$ and $z \in \mathbb{G}$, we get the claim. Thus, Corollary 6.15 is proved. \square

7 Proof of Theorem 1.3

We return now to the representation (4.6) which implies that

$$s_n(z) = \frac{1}{n} \sum_{j=1}^n \mathbf{E} R_{jj} = s_y(z) + \mathbf{E} \Lambda_n = s_y(z) + \mathbf{E} \frac{T_n(z)}{z + y(s(z) + m_n(z)) + \frac{y-1}{z}}. \quad (7.1)$$

We may continue the last equality as follows

$$s_n(z) = s_y(z) + \mathbf{E} \frac{\frac{1}{n} \sum_{j=1}^n \varepsilon_{j3} R_{jj}}{z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}} + \mathbf{E} \frac{\widehat{T}_n(z)}{z + y(s(z) + m_n(z)) + \frac{y-1}{z}}, \quad (7.2)$$

where

$$\widehat{T}_n = \frac{1}{n} \sum_{j=1}^n (\varepsilon_{j1} + \varepsilon_{j2}) R_{jj}.$$

Note that the definition of ε_{j3} in (4.6) and equality (8.59) in the Appendix together imply

$$\frac{1}{n} \sum_{j=1}^n \varepsilon_{j3} R_{jj} = \frac{y}{2n} (-m'_n(z) + \frac{m_n(z)}{z}). \quad (7.3)$$

Thus we may rewrite (7.2) as

$$\begin{aligned} s_n(z) = s_y(z) - \frac{y}{2n} \mathbf{E} \frac{m'_n(z) - \frac{m_n(z)}{z}}{z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}} \\ + \mathbf{E} \frac{\widehat{T}_n(z)}{z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}}. \end{aligned} \quad (7.4)$$

Denote by

$$\mathfrak{T} = \mathbf{E} \frac{\widehat{T}_n(z)}{z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}}.$$

7.1 Estimation of \mathfrak{T}

We represent \mathfrak{T}

$$\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2,$$

where

$$\begin{aligned} \mathfrak{T}_1 &= -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(\varepsilon_{j1} + \varepsilon_{j2}) \frac{1}{z + y m_n^{(j)}(z) + \frac{y-1}{z}}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_2 &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(\varepsilon_{j1} + \varepsilon_{j2}) (R_{jj} + \frac{1}{z + y m_n^{(j)}(z) + \frac{y-1}{z}})}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}. \end{aligned}$$

7.1.1 Estimation of \mathfrak{T}_1

We may decompose \mathfrak{T}_1 as

$$\mathfrak{T}_1 = \mathfrak{T}_{11} + \mathfrak{T}_{12}, \quad (7.5)$$

where

$$\begin{aligned} \mathfrak{T}_{11} &= -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(\varepsilon_{j1} + \varepsilon_{j2}) \frac{1}{z + y m_n^{(j)}(z) + \frac{y-1}{z}}}{z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{12} &= -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(\varepsilon_{j1} + \varepsilon_{j2}) \widetilde{\varepsilon}_{j3} \frac{1}{z + y m_n^{(j)}(z) + \frac{y-1}{z}}}{(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})}, \end{aligned}$$

where

$$\widetilde{\varepsilon}_{j3} = y\varepsilon_{j3} - \frac{1}{nz}.$$

It is easy to see that, by conditional expectation

$$\mathfrak{T}_{11} = 0. \quad (7.6)$$

Applying the Cauchy–Schwartz inequality, we get, for $\nu = 1, 2$,

$$\begin{aligned} & \left| \mathbf{E} \frac{\varepsilon_{j\nu} \tilde{\varepsilon}_{j3} \frac{1}{z + ym_n^{(j)}(z) + \frac{y-1}{z}}}{(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})} \right| \\ & \leq \mathbf{E}^{\frac{1}{2}} \left| \frac{\varepsilon_{j\nu}}{(z + ym_n^{(j)}(z) + \frac{y-1}{z})(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})} \right|^2 \end{aligned} \quad (7.7)$$

$$\times \mathbf{E}^{\frac{1}{2}} \left| \frac{\tilde{\varepsilon}_{j3}}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}} \right|^2. \quad (7.8)$$

Applying the Cauchy – Schwartz inequality again, we get

$$\begin{aligned} & \mathbf{E}^{\frac{1}{2}} \left| \frac{\varepsilon_{j\nu}}{(z + ym_n^{(j)}(z) + \frac{y-1}{z})(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})} \right|^2 \\ & \leq \mathbf{E}^{\frac{1}{4}} \frac{|\varepsilon_{j\nu}|^4}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^4} \mathbf{E}^{\frac{1}{4}} \frac{1}{|z + ym_n^{(j)}(z) + \frac{y-1}{z}|^4}. \end{aligned} \quad (7.9)$$

Inequalities (7.7), (7.9), Corollary 6.15 together imply

$$\begin{aligned} & \left| \mathbf{E} \frac{\varepsilon_{j\nu} \tilde{\varepsilon}_{j3} \frac{1}{z + ym_n^{(j)}(z) + \frac{y-1}{z}}}{(z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z})(z + y(m_n(z) + s(z)) + \frac{y-1}{z})} \right| \\ & \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{4}} \frac{|\varepsilon_{j\nu}|^4}{|z + ym_n^{(j)}(z) + \frac{y-1}{z} + s(z)|^4}. \end{aligned} \quad (7.10)$$

By Corollary 8.14, inequality (8.35) with $\alpha = 0$ and $\beta = 4$ in the Appendix we have for $\nu = 1, 2$

$$\mathbf{E}^{\frac{1}{4}} \frac{|\varepsilon_{j\nu}|^4}{|z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}|^4} \leq \frac{C}{\sqrt{nv} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}} \quad (7.11)$$

with some constant $C > 0$ depending on μ_4 and D only. We get from (7.10), and (7.11) that for $z \in \mathbb{G}$,

$$|\mathfrak{T}_1| \leq \frac{C}{(nv)^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}} \quad (7.12)$$

with some constant $C > 0$ depending on μ_4 and D only.

7.1.2 Estimation of \mathfrak{T}_2

Using the representation (4.6), we rewrite \mathfrak{T}_2 in the form

$$\mathfrak{T}_2 = \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(\varepsilon_{j1} + \varepsilon_{j2})^2 R_{jj}}{(z + ym_n^{(j)}(z) + \frac{y-1}{z})(z + y(s_y(z) + m_n(z)) + \frac{y-1}{z})}.$$

We decompose \mathfrak{T}_2 as follows

$$\mathfrak{T}_2 = \mathfrak{T}_{21} + \mathfrak{T}_{22}, \quad (7.13)$$

where

$$\begin{aligned} \mathfrak{T}_{21} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 R_{jj}}{(z + ym^{(j)}(z) + \frac{y-1}{z})(z + y(s_y(z) + m_n(z)) + \frac{y-1}{z})}, \\ \mathfrak{T}_{22} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{(\varepsilon_{j1}^2 + 2\varepsilon_{j1}\varepsilon_{j2}) R_{jj}}{(z + ym^{(j)}(z) + \frac{y-1}{z})(z + y(s_y(z) + m_n(z)) + \frac{y-1}{z})}. \end{aligned}$$

Applying the Cauchy – Schwartz inequality and inequality (8.43) in the Appendix, we obtain, for $z \in \mathbb{G}$,

$$|\mathfrak{T}_{22}| \leq \frac{C}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j1}|^2 |\varepsilon_{j1} + 2\varepsilon_{j2}|^2}{|z + ym^{(j)}(z) + \frac{y-1}{z}|^2 |z + y(s_y(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|^2} \mathbf{E}^{\frac{1}{2}} |R_{jj}|^2. \quad (7.14)$$

We note that

$$\mathbf{E}\{|\varepsilon_{j1}|^2 |\varepsilon_{j1} + 2\varepsilon_{j2}|^2 \big| \mathfrak{M}^{(j)}\} \leq C(\mathbf{E}^{\frac{1}{2}}\{|\varepsilon_{j1}|^4 \big| \mathfrak{M}^{(j)}\})(\mathbf{E}^{\frac{1}{2}}\{|\varepsilon_{j1}|^4 \big| \mathfrak{M}^{(j)}\} + \mathbf{E}^{\frac{1}{2}}\{|\varepsilon_{j2}|^4 \big| \mathfrak{M}^{(j)}\}).$$

Using Lemmas 8.12, 8.13, we get

$$\mathbf{E}\{|\varepsilon_{j1}|^2 |\varepsilon_{j1} + 2\varepsilon_{j2}|^2 \big| \mathfrak{M}^{(j)}\} \leq \frac{C}{n^2} + \frac{C}{n^2 v} (\operatorname{Im} m_n^{(j)}(z) + \frac{(1-y)v}{|z|^2}).$$

This implies that

$$\begin{aligned} |\mathfrak{T}_{22}| &\leq \left(\frac{C}{n|(z^2 + \frac{y-1}{z})^2 - 4y|^{\frac{1}{2}}} + \frac{C}{n\sqrt{v} |(z^2 + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}} \right) \\ &\quad \times \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \frac{1}{|z + ym_n^{(j)}(z) + \frac{y-1}{z}|^2} \mathbf{E} |R_{jj}|^2. \end{aligned}$$

Applying Corollary 6.15, we get

$$|\mathfrak{T}_{22}| \leq \frac{C}{nv^{\frac{3}{4}}}. \quad (7.15)$$

We continue now with \mathfrak{T}_{21} . We represent it in the form

$$\mathfrak{T}_{21} = H_1 + H_2, \quad (7.16)$$

where

$$H_1 = -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2}{(z + ym^{(j)}(z) + \frac{y-1}{z})^2 (z + y(s(z) + m_n(z)) + \frac{y-1}{z})},$$

$$H_2 = \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 (R_{jj} + \frac{1}{z + ym_n^{(j)}(z) + \frac{y-1}{z}})}{(z + ym^{(j)}(z) + \frac{y-1}{z}) (z + y(s(z) + m_n(z)) + \frac{y-1}{z})}.$$

Furthermore, using the representation

$$R_{jj} = -\frac{1}{z + ym_n^{(j)}(z) + \frac{y-1}{z}} + \frac{1}{z + ym_n^{(j)}(z) + \frac{y-1}{z}} (\varepsilon_{j1} + \varepsilon_{j2}) R_{jj}$$

(compare with (6.32)), we bound H_2 in the following way

$$|H_2| \leq H_{21} + H_{22},$$

where

$$H_{21} = \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{4|\varepsilon_{j1}|^3 |R_{jj}|}{|z + ym^{(j)}(z) + \frac{y-1}{z}|^2 |z + y(s(z) + m_n(z)) + \frac{y-1}{z}|},$$

$$H_{22} = \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{2|\varepsilon_{j2}|^3 |R_{jj}|}{|z + ym^{(j)}(z) + \frac{y-1}{z}|^2 |z + y(s(z) + m_n(z)) + \frac{y-1}{z}|}.$$

Using inequality (8.43) in the Appendix and Hölder inequality, we get, for

$\nu = 1, 2$

$$\begin{aligned}
H_{2\nu} &\leq \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{4|\varepsilon_{j\nu}|^3 |R_{jj}|}{|z + ym^{(j)}(z) + \frac{y-1}{z}|^2 |z + y(s(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|} \\
&\quad + \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{4|\varepsilon_{j\nu}|^3 |R_{jj}| |\varepsilon_{j3}|}{|z + ym^{(j)}(z) + \frac{y-1}{z}|^2 |z + y(s(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|} \\
&\quad \times \frac{1}{|z + y(s(z) + m_n(z)) + \frac{y-1}{z}|} \\
&\leq \frac{C}{n} \sum_{j=1}^n \mathbf{E}^{\frac{3}{4}} \frac{|\varepsilon_{j\nu}|^4}{|z + ym^{(j)}(z) + \frac{y-1}{z}|^{\frac{8}{3}} |z + y(s(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|^{\frac{4}{3}}} \mathbf{E}^{\frac{1}{4}} |R_{jj}|^4.
\end{aligned} \tag{7.17}$$

Applying Corollary 8.14 with $\beta = \frac{4}{3}$ and $\alpha = \frac{8}{3}$, we obtain, for $z \in \mathbb{G}$, and for $\nu = 1, 2$

$$\mathbf{E}^{\frac{3}{4}} \frac{|\varepsilon_{j2}|^4}{|z + ym^{(j)}(z) + \frac{y-1}{z}|^{\frac{8}{3}} |z + y(s(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|^{\frac{4}{3}}} \leq \frac{C}{(nv)^{\frac{3}{2}}}.$$

This yields together with Corollary 6.15 and inequality (7.17)

$$H_2 \leq \frac{C}{(nv)^{\frac{3}{2}}}. \tag{7.18}$$

Consider now H_1 . Using the equality

$$\begin{aligned}
\frac{1}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}} &= \frac{1}{z + 2s_y(z) + \frac{y-1}{z}} \\
&\quad - \frac{\Lambda_n(z)}{(z + 2s_y(z) + \frac{y-1}{z})(z + y(m_n(z) + s(z)) + \frac{y-1}{z})}
\end{aligned}$$

and

$$\Lambda_n = \Lambda_n^{(j)} + \tilde{\varepsilon}_{j3}, \tag{7.19}$$

we represent it in the form

$$H_1 = H_{11} + H_{12} + H_{13}, \tag{7.20}$$

where

$$\begin{aligned}
H_{11} &= -\frac{1}{(z + ys(z) + \frac{y-1}{z})^2} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2}{z + y(s(z) + m_n(z)) + \frac{y-1}{z}} \\
&= -s_y^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2}{z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}}, \\
H_{12} &= -\frac{1}{(z + ys_y(z) + \frac{y-1}{z})} \\
&\quad \times \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 \Lambda_n^{(j)}}{(z + ym_n^{(j)}(z) + \frac{y-1}{z})^2 (z + y(s_y(z) + m_n(z)) + \frac{y-1}{z})}, \\
H_{13} &= -\frac{1}{(z + ys_y(z) + \frac{y-1}{z})^2} \\
&\quad \times \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 \Lambda_n^{(j)}}{(z + ym_n^{(j)}(z) + \frac{y-1}{z})(z + y(s_y(z) + m_n(z)) + \frac{y-1}{z})}.
\end{aligned}$$

In order to apply conditional independence, we write

$$H_{11} = H_{111} + H_{112},$$

where

$$\begin{aligned}
H_{111} &= -s_y^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2}{z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}}, \\
H_{112} &= \frac{s_y^2(z)}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j2}^2 \varepsilon_{j3}}{(z + y(s_y(z) + m_n(z)) + \frac{y-1}{z})(z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z})}.
\end{aligned}$$

It is straightforward to check that

$$\mathbf{E}\{\varepsilon_{j2}^2 | \mathfrak{M}^{(j)}\} = \frac{1}{n^2} \sum_{l,k=1}^p (R_{l+n,k+n}^{(j)})^2 - \frac{1}{n^2} \sum_{l=1}^p (R_{l+n,l+n}^{(j)})^2.$$

By representation (4.4), we have

$$\begin{aligned}
\mathbf{E}\{\varepsilon_{j2}^2 | \mathfrak{M}^{(j)}\} &= \frac{y^2 z^2}{n^2} \text{Tr}(\mathbf{X}^* \mathbf{X} - z^2 \mathbf{I})^{-2} - \frac{y^2}{n^2} \sum_{l=1}^p (R_{l+n,l+n}^{(j)})^2 \\
&= \frac{y^2 z^2}{n^2} \sum_{k \in \mathbb{T}_j} \frac{1}{((s_k^{(j)})^2 - z^2)^2} - \frac{y^2}{n^2} \sum_{l=1}^p (R_{l+n,l+n}^{(j)})^2. \quad (7.21)
\end{aligned}$$

Applying equality (4.7), we get

$$(m_n(j)(z))' = \frac{1}{n} \sum_{k \in \mathbb{T}_j} \frac{1}{((s_k^{(j)})^2 - z^2)} - \frac{2z^2}{n} \sum_{k \in \mathbb{T}_j} \frac{1}{((s_k^{(j)})^2 - z^2)^2}. \quad (7.22)$$

Combining the last two equalities, we arrive

$$\mathbf{E}\{\varepsilon_{j2}^2 | \mathfrak{M}^{(j)}\} = -\frac{y^2}{2n} (m_n^{(j)}(z))' + \frac{1}{nz} m_n^{(j)}(z) - \frac{y^2}{n^2} \sum_{l=1}^p (R_{l+n, l+n}^{(j)})^2.$$

Using equality (7.3) for $m'_n(z)$ and the corresponding relation for $m_n^{(j)'}(z)$, we may write

$$H_{111} = L_1 + \cdots + L_5,$$

where

$$\begin{aligned} L_1 &= y^2 s^2(z) \frac{1}{2n} \mathbf{E} \frac{m'_n(z) - \frac{m_n(z)}{z}}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}} \\ L_2 &= -y^2 s^2(z) \frac{m_n^{(j)}(z) - m_n(z)}{nz(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})}, \\ L_3 &= y^2 s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\frac{1}{n^2} \sum_{l=1}^p (R_{l+n, l+n}^{(j)})^2}{z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}}, \\ L_4 &= y^2 s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\frac{1}{n} ((m_n^{(j)}(z))' - m'_n(z))}{z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}}, \\ L_5 &= y^2 s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\frac{1}{n} ((m_n^{(j)}(z))' - m'_n(z)) \tilde{\varepsilon}_{j3}}{(z + y(m_n(z) + s(z)) + \frac{y-1}{z})(z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z})}, \\ L_6 &= -y^2 s^2(z) \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\frac{1}{n} (m_n^{(j)}(z) - m_n(z)) \tilde{\varepsilon}_{j3}}{z(z + y(m_n(z) + s(z)) + \frac{y-1}{z})(z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z})}. \end{aligned}$$

First we note that

$$|m_n(z) - m_n^{(j)}(z)| \leq \frac{C}{nv}.$$

This inequality together with Lemma 8.11, inequality (8.30), imply that

$$|L_2| \leq \frac{C}{n^2 v^2 \sqrt{|(z + \frac{y-1}{z})^2 - 4y|}}$$

Using Lemma 8.11, inequality (8.30), 8.15, and Corollary 6.15, it is straightforward to check that

$$\begin{aligned} |L_3| &\leq \frac{C}{n\sqrt{|\left(z + \frac{y-1}{z}\right)^2 - 4y|}}, \\ |L_4| &\leq \frac{C}{n^2v^2\sqrt{|\left(z + \frac{y-1}{z}\right)^2 - 4y|}}, \\ |L_5| &\leq \frac{C}{n^3v^3|\left(z + \frac{y-1}{z}\right)^2 - 4y|}, \\ |L_6| &\leq \frac{C}{n^3v^3|\left(z + \frac{y-1}{z}\right)^2 - 4y|}. \end{aligned}$$

Applying inequality (8.43), we may write

$$|H_{12}| \leq \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j2}|^2 |\Lambda_n^{(j)}|}{|z + ym_n^{(j)}(z) + \frac{y-1}{z}| |z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}|}.$$

Conditioning on $\mathfrak{M}^{(j)}$ and applying Lemma 8.12, Lemma 8.11, inequality (8.30), Corollary 6.15 and equality (7.19), we get

$$\begin{aligned} |H_{12}| &\leq \frac{C}{nv} \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{|\Lambda_n^{(j)}|}{|z + ym_n^{(j)}(z) + \frac{y-1}{z}|} \\ &\leq \frac{C}{nv} \frac{1}{n} \sum_{j=1}^n (\mathbf{E} \frac{|\Lambda_n|}{|z + ym_n^{(j)}(z) + \frac{y-1}{z}|} + \mathbf{E} \frac{|\Lambda_n - \Lambda_n^{(j)}|}{|z + ym_n^{(j)}(z) + \frac{y-1}{z}|}) \\ &\leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2v^2}. \end{aligned}$$

By Lemma 8.17, we get

$$|H_{12}| \leq \frac{C}{n^2v^2}. \quad (7.23)$$

Similar we get

$$|H_{13}| \leq \frac{C}{n^2v^2}. \quad (7.24)$$

We rewrite now the equations (7.2) and (7.4) as follows,

$$\mathbf{E}\Lambda_n(z) = \mathbf{E}m_n(z) - s(z) = -\frac{y(1 - ys_y^2(z))}{2n} \mathbf{E} \frac{m'_n(z) - \frac{m_n(z)}{z}}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}} + \mathfrak{T}_3, \quad (7.25)$$

where

$$|\mathfrak{T}_3| \leq \frac{C}{n\sqrt{v}\sqrt{|(z + \frac{y-1}{z})^2 - 4y|}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}.$$

We use here inequalities (7.12), (7.15), (7.18), (7.23), (7.24) to bound $|\mathfrak{T}_3|$. Note that

$$1 - ys_y^2(z) = -s_y(z)\sqrt{(z + \frac{y-1}{z})^2 - 4y}.$$

In (7.25) we estimate now the remaining quantity

$$\mathfrak{T}_4 = -\frac{ys_y(z)\sqrt{(z + \frac{y-1}{z})^2 - 4y}}{2n}\mathbf{E}\frac{m'_n(z) - \frac{m_n(z)}{z}}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}}.$$

7.2 Estimation of \mathfrak{T}_4

Using that $\Lambda_n = m_n(z) - s(z)$ we rewrite \mathfrak{T}_4 as

$$\mathfrak{T}_4 = \mathfrak{T}_{41} + \cdots + \mathfrak{T}_{45},$$

where

$$\begin{aligned}\mathfrak{T}_{41} &= -\frac{ys_y(z)(s'_y(z) - \frac{s_y(z)}{z})}{2n}, \\ \mathfrak{T}_{42} &= \frac{ys_y(z)\sqrt{(z + \frac{y-1}{z})^2 - 4y}}{2n}\mathbf{E}\frac{m'_n(z) - s'_y(z)}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{43} &= \frac{ys_y(z)}{2n}\mathbf{E}\frac{(m'_n(z) - s'_y(z))\Lambda_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{44} &= \frac{ys_y(z)\sqrt{(z + \frac{y-1}{z})^2 - 4y}}{2n}\mathbf{E}\frac{m_n(z) - s_y(z)}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{45} &= \frac{ys_y(z)}{2n}\mathbf{E}\frac{(m_n(z) - s_y(z))\Lambda_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}.\end{aligned}$$

7.2.1 Estimation of \mathfrak{T}_{42}

First we investigate $m'_n(z)$. The following equality holds

$$m'_n(z) = -\frac{1}{n}\sum_{j=1}^n R_{jj}^2 = \frac{1}{n}\sum_{j=1}^n (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)})R_{jj} = \frac{s_y(z)}{n}\sum_{j=1}^n (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)}) + D_1, \quad (7.26)$$

where

$$\begin{aligned} D_1 &= \frac{1}{n} \sum_{j=1}^n (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)})(R_{jj} - s(z)) \\ &= \sum_{j=1}^n (\varepsilon_{j3} + \frac{1}{pz})(R_{jj} - s_y(z)). \end{aligned} \quad (7.27)$$

Using equality (6.40), we may write

$$\begin{aligned} m'_n(z) &= \frac{s_y(z)}{n} \sum_{j=1}^n \mathbf{E}(1 + \frac{1}{p} \sum_{l,k=1}^p X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{l+n,k+n}) R_{jj} + D_1 \\ &= \frac{s_y^2(z)}{n} \sum_{j=1}^n \mathbf{E}(1 + \frac{1}{p} \sum_{l,k=1}^p X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{l+n,k+n}) + D_1 + D_2, \end{aligned}$$

where

$$D_2 = \frac{s_y(z)}{n} \sum_{j=1}^n \mathbf{E}(1 + \frac{1}{p} \sum_{l,k=1}^p X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{l+n,k+n})(R_{jj} - s_y(z))$$

Denote by

$$\begin{aligned} \beta_{j1} &= \frac{1}{p} \sum_{l=1}^p \mathbf{E}[(R^{(j)})^2]_{l+n,l+n} - \frac{1}{p} \sum_{l=1}^p [(R)^2]_{l+n,l+n} = \frac{1}{p} \sum_{l \in \mathbb{T}_j} \mathbf{E}[(R^{(j)})^2]_{ll} - y m'_n(z) \\ &= \frac{1}{p} \frac{d}{dz} (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)}), \\ \beta_{j2} &= \frac{1}{p} \sum_{l=1}^p \mathbf{E}(X_{jl}^2 - 1) [(R^{(j)})^2]_{l+n,l+n}, \\ \beta_{j3} &= \frac{1}{p} \sum_{1 \leq l \neq k \leq p} \mathbf{E} X_{jl} X_{jk} [(R^{(j)})^2]_{l+n,k+n}. \end{aligned}$$

Using these notation we may write

$$m'_n(z) = s_y^2(z) \left(1 + y m'_n(z) - \frac{1-y}{z^2}\right) + \frac{s_y^2(z)}{n} \sum_{j=1}^n (\beta_{j1} + \beta_{j2} + \beta_{j3}) + D_1 + D_2.$$

Solving this equation with respect to $m'_n(z)$ we obtain

$$m'_n(z) = \frac{s_y^2(z)(1 - \frac{1-y}{z^2})}{1 - y s_y^2(z)} + \frac{1}{1 - s_y^2(z)} (D_3 + D_1 + D_2), \quad (7.28)$$

where

$$D_3 = \frac{s_y^2(z)}{n} \sum_{j=1}^n (\beta_{j1} + \beta_{j2} + \beta_{j3}).$$

Note that for the Marchenko – Pastur law

$$\frac{s_y^2(z)(1 - \frac{1-y}{z^2})}{1 - ys_y^2(z)} = -\frac{s_y(z)(1 - \frac{1-y}{z^2})}{z + 2ys_y(z) + \frac{y-1}{z}} = s'_y(z).$$

Applying this relation we rewrite equality (7.28) as

$$m'_n(z) - s'(z) = -\frac{1}{s_y(z)(z + 2s_y(z) + \frac{y-1}{z})}(D_1 + D_2 + D_3). \quad (7.29)$$

Using the last equality, we may represent \mathfrak{T}_{42} now as follows

$$\mathfrak{T}_{42} = \mathfrak{T}_{421} + \mathfrak{T}_{422} + \mathfrak{T}_{423},$$

where

$$\begin{aligned} \mathfrak{T}_{421} &= \frac{1}{n} \mathbf{E} \frac{D_1}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{422} &= \frac{1}{n} \mathbf{E} \frac{D_2}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{423} &= \frac{1}{n} \mathbf{E} \frac{D_3}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}. \end{aligned}$$

Recall that, by (7.27),

$$\mathfrak{T}_{421} = \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j3}(R_{jj} - s(z))}{(z + y(s(z) + m_n(z)) + \frac{y-1}{z})}. \quad (7.30)$$

Applying the Cauchy – Schwartz inequality, we get for $z \in \mathbb{G}$,

$$|\mathfrak{T}_{421}| \leq \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} |R_{jj} - s_y(z)|^2 \mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j3}|^2}{|z + y(s(z) + m_n(z)) + \frac{y-1}{z}|^2}.$$

Using Corollary 8.14, inequality (8.52) and Corollary 6.15, we get

$$|\mathfrak{T}_{421}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}}}. \quad (7.31)$$

7.2.2 Estimation of \mathfrak{T}_{423}

We represent now \mathfrak{T}_{423} in the form

$$\mathfrak{T}_{423} = \mathfrak{T}_{51} + \mathfrak{T}_{52} + \mathfrak{T}_{53}, \quad (7.32)$$

where

$$\mathfrak{T}_{5\nu} = \frac{1}{n^2} \sum_{j=1}^n \mathbf{E} \frac{\beta_{j\nu}}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}}, \quad \text{for } \nu = 1, 2, 3.$$

We consider the quantity $\mathfrak{T}_{5\nu}$, for $\nu = 1, 2, 3$. Applying the Cauchy-Schwartz inequality and inequality (8.43) in the Appendix as well, we get

$$|\mathfrak{T}_{5\nu}| \leq \frac{C}{n^2} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \frac{|\beta_{j\nu}|^2}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^2}.$$

By Lemma 8.19 together with Lemma 8.11 in the Appendix, we obtain

$$\mathbf{E}^{\frac{1}{2}} \frac{|\beta_{j\nu}|^2}{|z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}|^2} \leq \frac{C}{n^{\frac{1}{2}} v^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}.$$

This implies that

$$|\mathfrak{T}_{5\nu}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}. \quad (7.33)$$

Equality (7.32) and inequality (7.33) yield

$$|\mathfrak{T}_{423}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}. \quad (7.34)$$

7.2.3 Estimation of \mathfrak{T}_{422}

By the definitions of \mathfrak{T}_{423} and D_2 , we have

$$\mathfrak{T}_{422} = \frac{s_y(z)}{n^2} \sum_{j=1}^n \mathbf{E} \frac{(1 + \frac{1}{p} \sum_{l,k=1}^p X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{l+n, k+n})(R_{jj} - s_y(z))}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}.$$

We have

$$1 + \frac{1}{p} \sum_{l,k=1}^p X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{l+n, k+n} = 1 + y m'_n(z) + \beta_{j1} + \beta_{j2} + \beta_{j3}.$$

This implies

$$\mathfrak{T}_{422} = \mathfrak{T}_{60} + \cdots + \mathfrak{T}_{63},$$

where

$$\begin{aligned}\mathfrak{T}_{60} &= \frac{s_y(z)}{n} \mathbf{E} \frac{\Lambda_n(1 + ym'_n(z))}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{6\nu} &= \frac{s_y(z)}{n^2} \sum_{j=1}^n \mathbf{E} \frac{\beta_{j\nu}(R_{jj} - s_y(z))}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \text{ for } \nu = 1, 2, 3.\end{aligned}$$

Applying the Cauchy – Schwartz inequality, we get

$$|\mathfrak{T}_{60}| \leq \frac{C}{n\sqrt{y}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 \mathbf{E}^{\frac{1}{2}} \frac{|1 + m'_n(z)|^2}{|z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}|^2}$$

By inequality $|m'_n(z)| \leq v^{-1} \text{Im } m_n(z) + \frac{1-y}{|z|^2}$ and Lemma 8.11, we get

$$|\mathfrak{T}_{60}| \leq \frac{C}{nv\sqrt{y}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2.$$

Applying Lemma 8.17 below, we get

$$|\mathfrak{T}_{60}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}}}.$$

Similar to inequality (7.34), we get, for $\nu = 1, 2, 3$,

$$|\mathfrak{T}_{6\nu}| \leq \frac{C}{n^2 v^2 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}. \quad (7.35)$$

Combining (7.31), (7.34) and (7.35), we get, for $z \in \mathbb{G}$,

$$|\mathfrak{T}_{42}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}. \quad (7.36)$$

7.2.4 Estimation of \mathfrak{T}_{43}

Recall that

$$\mathfrak{T}_{43} = \frac{s(z)}{n} \mathbf{E} \frac{(m'_n(z) - s'_y(z))\Lambda_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}.$$

Applying equality (7.29), we obtain

$$\mathfrak{T}_{43} = \mathfrak{T}_{431} + \mathfrak{T}_{432} + \mathfrak{T}_{433},$$

where

$$\begin{aligned}\mathfrak{T}_{431} &= \frac{1}{2n(z + 2s_y(z) + \frac{y-1}{z})} \mathbf{E} \frac{D_1 \Lambda_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{432} &= \frac{1}{2n(z + 2s_y(z) + \frac{y-1}{z})} \mathbf{E} \frac{D_2 \Lambda_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{433} &= \frac{1}{2n(z + 2s(z) + \frac{y-1}{z})} \mathbf{E} \frac{D_3 \Lambda_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}.\end{aligned}$$

Applying the Cauchy – Schwartz inequality, we get

$$|\mathfrak{T}_{431}| \leq \frac{1}{n(z + 2s_y(z) + \frac{y-1}{z})} \mathbf{E}^{\frac{1}{2}} \frac{|D_1|^2}{|z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}|^2} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2.$$

By definition of D_1 and Lemma 8.17, we get

$$\begin{aligned}|\mathfrak{T}_{431}| &\leq \frac{C}{n^2 v |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{2}}} \sum_{j=1}^n \mathbf{E}^{\frac{1}{4}} \frac{|\varepsilon_{j3} + \frac{1}{pz}|^4}{|z + y(m_n(z) + s(z)) + \frac{y-1}{z}|^4} \\ &\quad \times \mathbf{E}^{\frac{1}{4}} |R_{jj} - s(z)|^4.\end{aligned}$$

Applying now Corollary 6.15 and Lemma 8.22, we get

$$|\mathfrak{T}_{431}| \leq \frac{4}{n^2 v^2 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{2}}}.$$

For $z \in \mathbb{G}$ this yields

$$|\mathfrak{T}_{431}| \leq \frac{4}{n^{\frac{3}{2}} v^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}.$$

Applying again the Cauchy – Schwartz inequality, we get for \mathfrak{T}_{432} accordingly

$$|\mathfrak{T}_{432}| \leq \frac{C}{n |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{2}}} \mathbf{E}^{\frac{1}{2}} |D_2|^2 \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2.$$

By Lemma 8.17, we have

$$|\mathfrak{T}_{432}| \leq \frac{C}{n^2 v |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{2}}} \mathbf{E}^{\frac{1}{2}} |D_2|^2. \quad (7.37)$$

By definition of D_2 ,

$$\mathbf{E}|D_2|^2 \leq \frac{1}{n} \sum_{j=1}^n (\mathbf{E}|\beta_{j1}|^2 + \mathbf{E}|\beta_{j2}|^2 + \mathbf{E}|\beta_{j3}|^2).$$

Applying Lemmas 8.19 with $\nu = 2, 3$, and 8.21, we get

$$\mathbf{E}|D_2|^2 \leq \frac{C}{n^2 v^4} + \frac{C}{nv^3}. \quad (7.38)$$

Inequalities (7.37) and (7.38) together imply, for $z \in \mathbb{G}$,

$$\begin{aligned} |\mathfrak{T}_{432}| &\leq \frac{C}{n^3 v^3 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{2}}} + \frac{C}{n^{\frac{5}{2}} v^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{2}}} \\ &\leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}. \end{aligned}$$

7.2.5 Estimation of $\mathfrak{T}_{44}, \mathfrak{T}_{45}$

Note that

$$\begin{aligned} \mathfrak{T}_{44} &= \frac{ys_y(z) \sqrt{(z + \frac{y-1}{z})^2 - 4}}{2nz} \mathbf{E} \frac{\Lambda_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}, \\ \mathfrak{T}_{45} &= \frac{ys_y(z)}{2nz} \mathbf{E} \frac{\Lambda_n^2}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}. \end{aligned}$$

Applying Lemma 8.11, inequality (8.30), we get

$$|\mathfrak{T}_{44}| \leq \frac{C}{2n|z|} \mathbf{E}|\Lambda_n|.$$

Therefore, by Lemma 8.17, we have

$$|\mathfrak{T}_{44}| \leq \frac{C}{2n^2 v^2}.$$

Using Lemma 8.11, inequality (8.30), Lemma 8.17, below, we get

$$|\mathfrak{T}_{45}| \leq \frac{C}{n^3 v^2 |z| \sqrt{(z + \frac{y-1}{z})^2 - 4y}}.$$

Applying now inequality (7.39) below, we get

$$|\mathfrak{T}_{45}| \leq \frac{C(y)}{n^3 v^{\frac{5}{2}}}.$$

7.2.6 Estimation of \mathfrak{T}_{41}

Finally we observe that

$$s'_y(z) = -\frac{s_y(z)(1 - \frac{y-1}{z^2})}{\sqrt{(z + \frac{y-1}{z})^2 - 4y}}$$

and

$$s'_y(z) - \frac{s_y(z)}{z} = -\frac{2s_y(z)(z + ys_y(z))}{z\sqrt{(z + \frac{y-1}{z})^2 - 4y}}.$$

Therefore

$$|\mathfrak{T}_{41}| \leq \frac{C}{n|(z^2 + y - 1)^2 - 4yz^2|^{\frac{1}{2}}}.$$

We have

$$(z^2 + y - 1)^2 - 4yz^2 = (z + \sqrt{y} - 1)(z + 1\sqrt{y} - 1)(z - \sqrt{y} + 1)(z - \sqrt{y} - 1).$$

For $z \in \mathbb{G}$ we get

$$|(z^2 + y - 1)^2 - 4yz^2|^{\frac{1}{2}} \geq C\sqrt{1 - \sqrt{y}\sqrt{v}}. \quad (7.39)$$

We may rewrite now

$$|\mathfrak{T}_{41}| \leq \frac{C(y)}{n\sqrt{v}}, \quad (7.40)$$

where

$$C(y) = \begin{cases} C, & \text{if } y = 1, \\ \frac{C}{\sqrt{1-\sqrt{y}}}, & \text{if } y < 1. \end{cases}$$

Combining now relations (7.25), (7.20), (7.18), (7.32), (7.36), (7.40), we get for $z \in \mathbb{G}$,

$$|\mathbf{E}\Lambda_n| \leq \frac{C}{nv^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v^{\frac{3}{2}}|(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}}. \quad (7.41)$$

The last inequality completes the proof of Theorem 1.3.

8 Appendix

8.1 Rosenthal's and Burkholder's Inequalities

In this subsection we state the Rosenthal and Burkholder inequalities starting with Rosenthal's inequality. Let ξ_1, \dots, ξ_n be independent random variables with $\mathbf{E}\xi_j = 0$, $\mathbf{E}\xi_j^2 = 1$ and for $p \geq 1$ $\mathbf{E}|\xi_j|^p \leq \mu_p$ for $j = 1, \dots, n$.

Lemma 8.1. (Rosenthal's inequality)

There exists an absolute constant C_1 such that

$$\mathbf{E} \left| \sum_{j=1}^n a_j \xi_j \right|^q \leq C_1^q q^q \left(\left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{q}{2}} + \mu_p \sum_{j=1}^n |a_j|^q \right)$$

Proof. For the proof of this inequality see [22] and [19]. \square

Let ξ_1, \dots, ξ_n be martingale-difference with respect to σ -algebras $\mathfrak{M}_j = \sigma(\xi_1, \dots, \xi_{j-1})$. Assume that $\mathbf{E}\xi_j^2 = 1$ and $\mathbf{E}|\xi_j|^q < \infty$, for $q \geq 2$.

Lemma 8.2. (Burkholder's inequality) *There exist an absolute constant C_2 such that*

$$\mathbf{E} \left| \sum_{j=1}^n \xi_j \right|^q \leq C_2^q q^q \left(\left(\mathbf{E} \left(\sum_{k=1}^n \mathbf{E} \{ \xi_k^2 | \mathfrak{M}_{k-1} \} \right)^{\frac{q}{2}} + \sum_{k=1}^p \mathbf{E} |\xi_k|^q \right) \right).$$

Proof. For the proof of this inequality see [5] and [18]. \square

We rewrite the Burkholder inequality for quadratic forms in independent random variables. Let ζ_1, \dots, ζ_n be independent random variables such that $\mathbf{E}\zeta_j = 0$, $\mathbf{E}|\eta_j|^2 = 1$ and $\mathbf{E}|\zeta_j|^q \leq \mu_q$. Let $a_{ij} = a_{ji}$ for all $i, j = 1, \dots, n$. Consider the quadratic form

$$Q = \sum_{1 \leq j \neq k \leq n} a_{jk} \zeta_j \zeta_k.$$

Lemma 8.3. *There exists an absolute constant C_2 such that*

$$\mathbf{E}|Q|^q \leq C_2^q \left(\mathbf{E} \left(\sum_{j=2}^n \left(\sum_{k=1}^{j-1} a_{jk} \zeta_k \right)^2 \right)^{\frac{q}{2}} + \mu_q \sum_{j=2}^n \mathbf{E} \left| \sum_{k=1}^{j-1} a_{jk} \zeta_k \right|^q \right).$$

Proof. Introduce the random variables

$$\xi_j = \zeta_j \sum_{k=1}^{j-1} a_{jk} \zeta_k, \quad j = 2, \dots, n.$$

It is straightforward to check that

$$\mathbf{E}\{\xi_j | \mathfrak{M}_{j-1}\} = 0,$$

and that ξ_j are \mathfrak{M}_j measurable. Hence ξ_1, \dots, ξ_n are martingale-differences. We may write

$$Q = 2 \sum_{j=2}^n \xi_j$$

Applying now Lemma 8.2 and using

$$\begin{aligned} \mathbf{E}\{|\xi_j|^2 | \mathfrak{M}_{j-1}\} &= \left(\sum_{k=1}^{j-1} a_{jk} \eta_k \right)^2 \mathbf{E}\zeta_j^2, \\ \mathbf{E}|\xi_j|^q &= \mathbf{E}|\eta_j|^q \mathbf{E} \left| \sum_{k=1}^{j-1} a_{jk} \zeta_j \right|^q, \end{aligned}$$

we get the claim. Thus, Lemma 8.3 is proved. \square

8.2 Auxiliary Inequalities for Resolvent Matrices

We shall use the following relation between resolvent matrices. Let \mathbb{A} and \mathbb{B} be two Hermitian matrices and let $\mathbf{R}_{\mathbf{A}} = (\mathbb{A} - z\mathbf{I})^{-1}$ and $\mathbf{R}_{\mathbf{B}} = (\mathbb{B} - z\mathbf{I})^{-1}$ denote their resolvent matrices. Recall the resolvent equality

$$\mathbf{R}_{\mathbf{A}} - \mathbf{R}_{\mathbf{B}} = \mathbf{R}_{\mathbf{A}}(\mathbf{B} - \mathbf{A})\mathbf{R}_{\mathbf{B}} = -\mathbf{R}_{\mathbf{B}}(\mathbf{B} - \mathbf{A})\mathbf{R}_{\mathbf{A}}. \quad (8.1)$$

Recall the equation, for $j \in \mathbb{T}_{\mathbb{J}}$, and $\mathbb{J} \subset \mathbb{T}$ (compare with (6.32))

$$R_{jj}^{(\mathbb{J})} = -\frac{1}{z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{nz}} + \frac{1}{z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{nz}} \varepsilon_j^{(\mathbb{J})} R_{jj}^{(\mathbb{J})}, \quad (8.2)$$

where $\varepsilon_j^{(\mathbb{J})} = \varepsilon_{j1}^{(\mathbb{J})} + \varepsilon_{j2}^{(\mathbb{J})} + \varepsilon_{j3}^{(\mathbb{J})}$ and

$$\begin{aligned}\varepsilon_{j1}^{(\mathbb{J})} &= \frac{1}{n} \sum_{l=1}^p (X_{jl}^2 - 1) R_{l+n, l+n}^{(\mathbb{J}, j)}, \\ \varepsilon_{j2}^{(\mathbb{J})} &= \frac{1}{n} \sum_{1 \leq l \neq k \leq p} X_{jl} X_{jk} R_{k+n, l+n}^{(\mathbb{J}, j)}, \\ \varepsilon_{j3}^{(\mathbb{J})} &= m_n^{(\mathbb{J})}(z) - m_n^{(\mathbb{J}, j)}(z).\end{aligned}$$

Summing these equations for $j \in \mathbb{T}_{\mathbb{J}}$, we get

$$m_n^{(\mathbb{J})}(z) = -\frac{n - |\mathbb{J}|}{n(z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{pz})} + \frac{T_n^{(\mathbb{J})}}{z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{pz}}, \quad (8.3)$$

where

$$T_n^{(\mathbb{J})} = \frac{1}{n} \sum_{j \in \mathbb{T}_{\mathbb{J}}} \varepsilon_j^{(\mathbb{J})} R_{jj}^{(\mathbb{J})}.$$

Note that

$$\begin{aligned}\frac{1}{z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{pz}} &= \frac{1}{z + ys_y(z) + \frac{p-n+|F|}{pz}} \\ &\quad - \frac{m_n^{(\mathbb{J})}(z) - s_y(z)}{(z + ys_y(z) + \frac{n-p+|\mathbb{J}|}{pz})(z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{pz})} \\ &= -s_y(z) + \frac{s_y(z)\Lambda_n^{(\mathbb{J})}(z)}{z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{pz}},\end{aligned} \quad (8.4)$$

where

$$\Lambda_n^{(\mathbb{J})} = \Lambda_n^{(\mathbb{J})}(z) = m_n^{(\mathbb{J})}(z) - s_y(z).$$

Equalities (8.3) and (8.4) together imply

$$\begin{aligned}\Lambda_n^{(\mathbb{J})} &= -\frac{s_y(z)\Lambda_n^{(\mathbb{J})}}{z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{pz}} \\ &\quad + \frac{T_n^{(\mathbb{J})}}{z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{pz}} + \frac{|\mathbb{J}|}{n(z + ym_n^{(\mathbb{J})}(z) + \frac{n-p+|\mathbb{J}|}{pz})}.\end{aligned}$$

Solving this with respect to $\Lambda_n^{(\mathbb{J})}$, we get

$$\Lambda_n^{(\mathbb{J})} = \frac{T_n^{(\mathbb{J})}}{z + y(m_n^{(\mathbb{J})}(z) + s_y(z)) + \frac{n-p+|\mathbb{J}|}{pz}} + \frac{|\mathbb{J}|}{n(z + y(m_n^{(\mathbb{J})}(z) + s_y(z)) + \frac{n-p+|\mathbb{J}|}{pz})}. \quad (8.5)$$

Lemma 8.4. *For any $z = u + iv$ with $v > 0$ and for any $\mathbb{J} \subset \mathbb{T}$, we have*

$$\frac{1}{n} \sum_{l=1}^p \sum_{k=1}^p |R_{l+n,k+n}^{(\mathbb{J})}|^2 \leq v^{-1} \text{Im} m_n^{(\mathbb{J})}(z) + \frac{1-y}{|z|^2}, \quad (8.6)$$

and, for any $\mathbb{J} \subset \mathbb{L} = \{1, \dots, p\}$

$$\frac{1}{n} \sum_{l=1}^n \sum_{k=1}^n |R_{l,k}^{(\mathbb{J})}|^2 \leq v^{-1} \text{Im} m_n^{(\mathbb{J})}(z) + \frac{1-y}{|z|^2}, \quad (8.7)$$

For any $\mathbb{J} \subset \mathbb{T}$, $l = 1, \dots, p$,

$$\sum_{k=1}^p |R_{l+n,k+n}^{(\mathbb{J})}|^2 \leq v^{-1} \text{Im} R_{l+n,l+n}^{(\mathbb{J})}. \quad (8.8)$$

and, for any $\mathbb{J} \subset \mathbb{L} = \{1, \dots, p\}$, $l = 1, \dots, n$,

$$\sum_{k=1}^n |R_{l,k}^{(\mathbb{J})}|^2 \leq v^{-1} \text{Im} R_{l,l}^{(\mathbb{J})}. \quad (8.9)$$

For any $\mathbb{J} \subset \mathbb{T}$, $l = 1, \dots, p$,

$$\sum_{k=1}^p |[(\mathbf{R}^{(\mathbb{J})})^2]_{l+n,k+n}|^2 \leq C v^{-3} \text{Im} R_{l+n,l+n}^{(\mathbb{J})}. \quad (8.10)$$

and, for any $\mathbb{J} \subset \mathbb{L} = \{1, \dots, p\}$, $l = 1, \dots, n$,

$$\sum_{k=1}^n |[(\mathbf{R}^{(\mathbb{J})})^2]_{l,k}|^2 \leq C v^{-3} \text{Im} R_{l,l}^{(\mathbb{J})}. \quad (8.11)$$

Moreover, for any $\mathbb{J} \subset T$ and for any $l \in \mathbb{T}_{\mathbb{J}}$ we have

$$\frac{1}{n} \sum_{l=1}^p |[(\mathbf{R}^{(\mathbb{J})})^2]_{l+n,l+n}|^2 \leq v^{-3} \text{Im} m_n^{(\mathbb{J})}(z), \quad (8.12)$$

and, for any $\mathbb{J} \subset \mathbb{L} = \{1, \dots, p\}$,

$$\frac{1}{n} \sum_{l=1}^n |[(R^{(\mathbb{J})})^2]_{l,l}|^2 \leq v^{-3} \text{Im} m_n^{(\mathbb{J})}(z). \quad (8.13)$$

For any $q \geq 1$,

$$\frac{1}{n} \sum_{l=1}^p |[(R^{(\mathbb{J})})^2]_{l+n, l+n}|^q \leq v^{-q} \frac{1}{n} \sum_{l=1}^p \text{Im}^q R_{l+n, l+n}^{(\mathbb{J})}, \quad (8.14)$$

and

$$\frac{1}{n} \sum_{l=1}^n |[(R^{(\mathbb{J})})^2]_{l, l}|^q \leq v^{-q} \frac{1}{n} \sum_{l=1}^n \text{Im}^q R_{l, l}^{(\mathbb{J})}. \quad (8.15)$$

Finally,

$$\frac{1}{n} \sum_{l, k=1}^p |[(R^{(\mathbb{J})})^2]_{l+n, k+n}|^2 \leq v^{-3} \text{Im} m_n^{(\mathbb{J})}(z), \quad (8.16)$$

and

$$\frac{1}{n} \sum_{l, k=1}^p |[(R^{(\mathbb{J})})^2]_{l+n, k+n}|^{2q} \leq v^{-3q} \frac{1}{n} \sum_{l=1}^p \text{Im}^q R_{l+n, l+n}^{(\mathbb{J})}, \quad (8.17)$$

We have as well

$$\frac{1}{n^2} \sum_{l, k=1}^p |[(R^{(\mathbb{J})})^2]_{l+n, k+n}|^{2p} \leq v^{-2q} \left(\frac{1}{n} \sum_{l \in \mathbb{T}_{\mathbb{J}}} \text{Im}^p R_{l+n, l+n}^{(\mathbb{J})} \right)^2. \quad (8.18)$$

Proof. Firstly we note that

$$\max \left\{ \frac{1}{n} \sum_{l=1}^p \sum_{k=1}^p |R_{l+n, k+n}^{(\mathbb{J})}|^2, \frac{1}{n} \sum_{l=1}^n \sum_{k=1}^n |R_{l, k}^{(\mathbb{J})}|^2 \right\} \leq \frac{1}{n} \text{Tr} |\mathbf{R}^{(\mathbb{J})}|^2. \quad (8.19)$$

Furthermore,

$$\frac{1}{n} \text{Tr} |R^{(\mathbb{J})}|^2 = v^{-1} \frac{1}{n} \text{Im} \text{Tr} \mathbf{R}^{(\mathbb{J})} = 2v^{-1} \text{Im} m_n^{(\mathbb{J})}(z) + \frac{1 - y - \frac{|\mathbb{J}|}{n}}{|z|^2}$$

These relations imply inequalities (8.6) and (8.7). Note that

$$\max \left\{ \sum_{k=1}^p |R_{l+n, k+n}^{(\mathbb{J})}|^2, \sum_{k=1}^n |R_{l, k}^{(\mathbb{J})}|^2 \right\} \leq \sum_k^* |R_{l, k}^{(\mathbb{J})}|^2.$$

Here we denote by \sum_k^* the sum over $k \in \mathbb{T}_{\mathbb{J}} \cup \{1, \dots, p\}$ if $\mathbb{J} \subset \mathbb{T}$, and the sum over $k \in \{1, \dots, n\} \cup \{\mathbb{L} \setminus \mathbb{J}\}$. Let us denote by $\lambda_k^{(\mathbb{J})}$ the eigenvalues of

the matrix $\mathbf{W}^{(\mathbb{J})}$. Let denote now by $\mathbf{u}_k^{(\mathbb{J})} = (u_{kl}^{(\mathbb{J})})$ the eigenvector of the matrix $\mathbf{W}^{(\mathbb{J})}$ corresponding to the eigenvalue $\lambda_k^{(\mathbb{J})}$. Using this notation we may write

$$R_{lk}^{(\mathbb{J})} = \sum_q^* \frac{1}{\lambda_q^{(\mathbb{J})} - z} u_{lq}^{(\mathbb{J})} u_{kq}^{(\mathbb{J})}. \quad (8.20)$$

It is straightforward to check that the following inequality holds

$$\begin{aligned} \sum_k^* |R_{kl}^{(\mathbb{J})}|^2 &\leq \sum_q^* \frac{1}{|\lambda_q^{(\mathbb{J})} - z|^2} |u_{lq}^{(\mathbb{J})}|^2 \\ &= v^{-1} \text{Im} \left(\sum_q^* \frac{1}{\lambda_q^{(\mathbb{J})} - z} |u_{lq}^{(\mathbb{J})}|^2 \right) = v^{-1} \text{Im} R_{ll}^{(\mathbb{J})}. \end{aligned}$$

Thus, inequalities (8.8) and (8.9) are proved. Similarly we get

$$\sum_k^* |[(R^{(\mathbb{J})})^2]_{kl}|^2 \leq \sum_q^* \frac{1}{|\lambda_q^{(\mathbb{J})} - z|^4} |u_{lq}^{(\mathbb{J})}|^2 \leq v^{-3} \text{Im} R_{ll}^{(\mathbb{J})}.$$

This implies inequalities (8.10) and (8.11). To prove inequality (8.12) and (8.13) we observe that

$$|[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}| \leq \sum_k^* |R_{lk}^{(\mathbb{J})}|^2. \quad (8.21)$$

This inequality implies

$$\frac{1}{n} \sum_l^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^2 \leq \frac{1}{n} \sum_l^* \left(\sum_k^* |R_{lk}^{(\mathbb{J})}|^2 \right)^2.$$

Applying now inequality (8.8), we get

$$\frac{1}{n} \sum_l^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^2 \leq v^{-2} \frac{1}{n} \sum_l^* \text{Im}^2 R_{ll}^{(\mathbb{J})}.$$

Using $|R_{ll}^{(\mathbb{J})}| \leq v^{-1}$ this leads to the following bound

$$\frac{1}{n} \sum_l^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^2 \leq v^{-3} \frac{1}{n} \sum_l^* \text{Im} R_{ll}^{(\mathbb{J})} = v^{-3} \text{Im} m_n^{(\mathbb{J})}(z). \quad (8.22)$$

Thus inequalities (8.12) and (8.13) are proved. Furthermore, applying inequality (8.21), we may write

$$\frac{1}{n} \sum_l^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{ll}|^4 \leq \frac{1}{n} \sum_l^* \left(\sum_k^* |R_{lk}^{(\mathbb{J})}|^2 \right)^4.$$

Applying (8.8), this inequality yields

$$\frac{1}{n} \sum_l |[(R^{(\mathbb{J})})^2]_{ll}|^4 \leq v^{-4} \frac{1}{n} \sum_l^* \text{Im}^4 R_{ll}^{(\mathbb{J})}.$$

The last inequality proves inequality (8.14). Similarly we get inequality (8.15). Note that

$$\begin{aligned} \frac{1}{n} \sum_{l,k}^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 &\leq \frac{1}{n} \text{Tr} |\mathbf{R}^{(\mathbb{J})}|^4 = \frac{1}{n} \sum_l^* \frac{1}{|\lambda_l^{(\mathbb{J})} - z|^4} \\ &\leq v^{-3} \text{Im} \frac{1}{n} \sum_l^* \frac{1}{\lambda_l^{(\mathbb{J})} - z} \leq v^{-3} (\text{Im} m_n^{(\mathbb{J})}(z) + \frac{1 - y - \frac{v|\mathbb{J}|}{n}}{|z|^2}). \end{aligned}$$

Thus, inequality (8.16) is proved. To finish we note that

$$\frac{1}{n} \sum_{l,k}^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^4 \leq \frac{1}{n} \sum_l^* (\sum_k^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2)^2.$$

Applying inequality (8.10), we get

$$\frac{1}{n} \sum_{l,k}^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^4 \leq v^{-6} \frac{1}{n} \sum_l^* (\text{Im} R_{ll}^{(\mathbb{J})})^2.$$

To prove inequality (8.18), we note

$$|[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^2 \leq (\sum_q^* |R_{lq}^{(\mathbb{J})}|^2) (\sum_q^* |R_{kq}^{(\mathbb{J})}|^2).$$

This inequality implies

$$\frac{1}{n^2} \sum_{l,k}^* |[(\mathbf{R}^{(\mathbb{J})})^2]_{lk}|^{2p} \leq (\frac{1}{n} \sum_{l,k \in}^* (\sum_q^* |R_{lq}^{(\mathbb{J})}|^2)^p)^2 (\text{Im} R_{ll}^{(\mathbb{J})})^2.$$

Applying inequality (8.6), we get the claim. Thus, Lemma 8.4 is proved. \square

Lemma 8.5. *For any $s \geq 1$, and for any $z = u + iv$ and for any $\mathbb{J} \subset \mathbb{T}$, and $l = 1, \dots, p$,*

$$|R_{jj}^{(\mathbb{J})}(u + iv/s)| \leq s |R_{l+n, l+n}^{(\mathbb{J})}(u + iv)|. \quad (8.23)$$

and

$$\left| \frac{1}{u + iv/s + ym_n(u + iv/s) + \frac{y-1}{u+iv/s}} \right| \leq s \left| \frac{1}{u + iv + ym_n(u + iv) + \frac{y-1}{u+iv}} \right|. \quad (8.24)$$

Proof. See Lemma 3.4 in [20]. For the readers convenience we include the short argument here. Note that, for any $l = 1, \dots, p$,

$$\left| \frac{d}{dv} \log R_{l+n, l+n}^{(\mathbb{J})}(u + iv) \right| \leq \frac{1}{|R_{l+n, l+n}^{(\mathbb{J})}(u + iv)|} \left| \frac{d}{dv} R_{l+n, l+n}^{(\mathbb{J})}(u + iv) \right|.$$

Furthermore,

$$\frac{d}{dv} R_{l+n, l+n}^{(\mathbb{J})}(u + iv) = [(\mathbf{R}^{(\mathbb{J})})^2]_{l+n, l+n}(u + iv)$$

and

$$|[(\mathbf{R}^{(\mathbb{J})})^2]_{l+n, l+n}(u + iv)| \leq v^{-1} \operatorname{Im} R_{l+n, l+n}^{(\mathbb{J})}.$$

From here it follows that

$$\left| \frac{d}{dv} \log R_{l+n, l+n}^{(\mathbb{J})}(u + iv) \right| \leq v^{-1}.$$

We may write now

$$|\log R_{l+n, l+n}^{(\mathbb{J})}(u + iv) - \log R_{l+n, l+n}^{(\mathbb{J})}(u + iv/s)| \leq \int_{v/s}^v \frac{du}{u} = \log s.$$

The last inequality yields (8.23). Similarly we have

$$\left| -\frac{d}{dz} \log \left\{ z + ym_n(z) + \frac{y-1}{z} \right\} \right| \leq \frac{|1 + y \frac{d}{dz} m_n(z) - \frac{1-y}{z^2}|}{|z + ym_n(z) + \frac{y-1}{z}|}.$$

Using that

$$\begin{aligned} |z + ym_n(z) + \frac{y-1}{z}| &\geq |\operatorname{Im} \{ z + ym_n(z) + \frac{y-1}{z} \}| = v(1 + v^{-1} y \operatorname{Im} m_n(z) + \frac{(1-y)v}{|z|^2}), \\ |1 + y \frac{d}{dz} m_n(z) - \frac{1-y}{z^2}| &\leq 1 + y \left| \frac{d}{dz} m_n(z) \right| + \frac{1-y}{|z|^2} \leq 1 + yv^{-1} \operatorname{Im} m_n(z) + \frac{(1-y)v}{|z|^2}. \end{aligned}$$

The last inequalities together imply

$$\left| -\frac{d}{dz} \log \left\{ z + ym_n(z) + \frac{y-1}{z} \right\} \right| \leq v^{-1}.$$

From here (8.24) follows. Thus Lemma 8.5 is proved. \square

8.3 Some Auxiliary Bounds for Resolvent Matrices for $z = u + iV$ with $V = 4\sqrt{y}$

We shall use the bound for the $\varepsilon_{j\nu}$ for $V = \sqrt{y}$.

Lemma 8.6. *Assuming the conditions of Theorem 1.1, we get*

$$\mathbf{E}|\varepsilon_{j2}|^q \leq \frac{C^q q}{n^{\frac{q}{2}}}.$$

Proof. Conditioning on $\mathfrak{M}^{(j)}$ and applying Burkholder's inequality (see Lemma 8.3), we get

$$\mathbf{E}|\varepsilon_{j2}|^q \leq C_2^q q^q n^{-q} (\mathbf{E} |\sum_{k=2}^p (\sum_{l=1}^{k-1} R_{k+n,l+n}^{(j)} X_{jk})^2|^{\frac{q}{2}} + \mu_q \sum_{k=2}^p \mathbf{E} |\sum_{l=1}^{k-1} R_{k+n,l+n}^{(j)} X_{jl}|^q).$$

Applying now Corollary 6.1 and Rosenthal's inequality, we get

$$\mathbf{E}|\varepsilon_{j2}|^q \leq C_2^q q^{2q} n^{-\frac{q}{2}} + \mu_q n^{-q} q^{2q} \sum_{l \in \mathbb{T}_j} \mathbf{E} (\sum_{k \in \mathbb{T}_j} |R_{kl}^{(j)}|^2)^{\frac{q}{2}} + \mu_q^2 n^{-q} q^{2q} \sum_{k,l=1}^p \mathbf{E} |R_{k+n,l+n}^{(j)}|^q.$$

Using that $|R_{k+n,l+n}^{(j)}| \leq \frac{1}{4}$ and $\sum_{l=1}^p |R_{kl}^{(j)}|^2 \leq \frac{1}{16}$ and $\mu_q \leq D^{\frac{q}{4}} n^{\frac{q}{4}-1} \mu_4$, we get

$$\mathbf{E}|\varepsilon_{j2}|^q \leq C_2^q q^{2q} n^{-\frac{q}{2}}.$$

Thus Lemma 8.6 is proved. \square

Lemma 8.7. *Assuming the conditions of Theorem 1.1, we get*

$$\mathbf{E}|\varepsilon_{j1}|^q \leq \frac{C^q q}{n^{\frac{q}{2}}}.$$

Proof. Conditioning and applying Rosenthal's inequality, we obtain

$$\mathbf{E}|\varepsilon_{j1}|^q \leq C^q q^q n^{-q} (\mu_4^{\frac{q}{2}} \mathbf{E} (\sum_{l=1}^p |R_{l+n,l+n}^{(j)}|^2)^{\frac{q}{2}} + \mu_{2q} \sum_{l=1}^p \mathbf{E} |R_{l+n,l+n}^{(j)}|^q).$$

Using that $|R_{l+n,l+n}^{(j)}| \leq \frac{1}{4}$ and $\mu_{2q} \leq D^{2q-4} n^{\frac{q}{2}-1} \mu_4$, we get

$$\mathbf{E}|\varepsilon_{j1}|^q \leq C^q q^q n^{-\frac{q}{2}}.$$

Thus Lemma 8.7 is proved. \square

Lemma 8.8. *Assuming the conditions of Theorem 1.1, we get, for any $q \geq 1$,*

$$|\varepsilon_{j3}|^q \leq \frac{C^q}{n^q}.$$

Proof. Recall that

$$\varepsilon_{j3} = \frac{1}{n} \sum_{l=1}^p R_{l+n, l+n} - \frac{1}{n} \sum_{l=1}^p R_{l+n, l+n}^{(j)}.$$

It is straightforward to check that

$$\frac{1}{n} \sum_{l=1}^p R_{l+n, l+n} - \frac{1}{n} \sum_{l=1}^p R_{l+n, l+n}^{(j)} = \frac{1}{2n} \text{Tr } \mathbf{R} - \frac{1}{2n} \text{Tr } \mathbf{R}^{(j)} + \frac{1}{2nz}$$

From here follows immediately the bound

$$|\varepsilon_{j3}| \leq \frac{1}{nv}, \text{ a. s.}$$

See for instance [9], Lemma 3.3. The last bound implies the claim. \square

8.4 Some Auxiliary Bounds for Resolvent Matrices for $z \in \mathbb{G}$

We start from the simple lemma on the behavior of Stieltjes transform of symmetrizing Marchenko – Pastur distribution with parameter y .

Lemma 8.9. *For all $z = u + iv \in \mathbb{C}_+$ with $1 - \sqrt{y} \leq |u| \leq 1 + \sqrt{y}$ and for any $0 < y \leq 1$, we have*

$$|z + y s_y(z)| \geq \frac{1}{1 + \sqrt{y}}.$$

Proof. We consider representation

$$s_y(z) = \frac{-(z + \frac{y-1}{z}) + \sqrt{(z + \frac{y-1}{z})^2 - 4y}}{2y}.$$

We may rewrite this equality as follows

$$s_y(z) = \frac{-(z + \frac{y-1}{z}) + \sqrt{(z + \frac{1-y}{z})^2 - 4}}{2y}.$$

Introduce the notation

$$w = z + \frac{1-y}{z}.$$

In these notation we may write

$$z + ys_y(z) = w + s(w),$$

where $s(w) = \frac{-w + \sqrt{w^2 - 4}}{2}$ denotes the Stieltjes transform of the semi-circular law. Here we choose the branch of the root such that $\text{Im} \sqrt{\cdot} \geq 0$. Furthermore, we note

$$\text{Im } w \begin{cases} \geq 0, & \text{if } |z| \geq \sqrt{1-y}, \\ < 0, & \text{if } |z| \leq \sqrt{1-y} \end{cases}.$$

Therefore, since for Stieltjes transform of the semi-circular law $|w + s(w)| \geq 1$ for any $w \in \mathbb{C}_+$, we get

$$|z + ys_y(z)| \geq 1, \text{ for all } |z| \geq \sqrt{1-y}.$$

Consider now the function $f(z) := z + ys_y(z)$ in the domain $\mathbb{K} := \{z \in \mathbb{C}_+ : |z| \leq \sqrt{1-y}\}$. This function is analytic in the open domain \mathbb{K} and continuous in the domain \mathbb{K} . By the Maximum Principle, the minimum of modulus of the function $f(z)$ in the domain \mathbb{K} is attained on its boundary. It is straightforward to check that $|f(z)| = 1$, for all $z \in \mathbb{C}_+ : |z| = \sqrt{1-y}$. Moreover, $|f(z)| = 1$, for $z = u$ with $-\sqrt{1-y} \leq u \leq -1 + \sqrt{y}$ or $1 - \sqrt{y} \leq u \leq \sqrt{1-y}$. Let $z = \pm(1 - \sqrt{y}) + iv$ with $|z| \leq \sqrt{1-y}$ and $v > 0$. We consider the case $z = 1 - \sqrt{y} + iv$ only. It is straightforward to check that

$$\text{Re}(w^2 - 4) \leq 0, \quad \text{Im}(w^2 - 4) \leq 0.$$

Moreover,

$$|w| \leq 2, \text{ and } |\sqrt{w^2 - 4}| \leq 2\sqrt{y}$$

We may write

$$f(z) = \frac{2}{w - \sqrt{w^2 - 4}}.$$

These relations together imply that, for $z = 1 - \sqrt{y} + iv$ with $|z| \leq \sqrt{1-y}$ and $v > 0$

$$|f(z)| \geq \frac{1}{1 + \sqrt{y}}.$$

This inequality proves the Lemma. □

Recall that

$$\begin{aligned} \mathbb{G} &:= \{z = u + iv \in \mathbb{C}^+ : u \in \mathbb{J}_\varepsilon, v \geq v_0/\sqrt{\gamma}\}, \text{ where } v_0 = A_0 n^{-1}, \quad (8.25) \\ \mathbb{J}_\varepsilon &= \{u : (1 - \sqrt{y}) + \varepsilon \leq |u| \leq 1 + \sqrt{y} - \varepsilon\}, \quad \varepsilon := c_1 n^{-\frac{2}{3}}, \\ \gamma &= \gamma(u) = \min\{|u| - 1 + \sqrt{y}, 1 + \sqrt{y} - |u|\}. \end{aligned}$$

In the next lemma we prove some simple inequalities for the region \mathbb{G} .

Lemma 8.10. *For any $z \in \mathbb{G}$ we have*

$$\begin{aligned} |(z + \frac{y-1}{z})^2 - 4y| &\geq \frac{4y}{5} \max\{\gamma, v\}, \\ nv \sqrt{|(z + \frac{y-1}{z})^2 - 4y|} &\geq A_0 \frac{4y}{5}. \end{aligned} \quad (8.26)$$

Proof. For $z \in \mathbb{G}$ we have

$$|z| \leq ((1 + \sqrt{y})^2 + v^2)^{\frac{1}{2}} \leq 5/\sqrt{y}. \quad (8.27)$$

Furthermore, we observe that

$$\begin{aligned} |(z + \frac{y-1}{z})^2 - 4y| &= |(z + \frac{y-1}{z} - 2\sqrt{y})(z + \frac{y-1}{z} + 2\sqrt{y})| \\ &= \frac{|z - (1 + \sqrt{y})|(z + (1 + \sqrt{y}))(z - (1 - \sqrt{y}))(z + (1 - \sqrt{y}))|}{|z|^2}. \end{aligned} \quad (8.28)$$

Assume that $1 - \sqrt{y} \leq u \leq 1$. Then, for $z \in \mathbb{G}$,

$$\begin{aligned} |(z + \frac{y-1}{z})^2 - 4y| &\geq v \frac{|u - (1 + \sqrt{y}))(u + (1 + \sqrt{y}))|(z + (1 - \sqrt{y}))|}{|z|^2} \\ &\geq 4v\sqrt{y} \frac{1}{|z|} \geq v \frac{4y}{5}, \end{aligned}$$

and

$$\begin{aligned} |(z + \frac{y-1}{z})^2 - 4y| &\geq \gamma \frac{|u - (1 + \sqrt{y}))(u + (1 + \sqrt{y}))(u + (1 - \sqrt{y}))|}{|z|^2} \\ &\geq \gamma \frac{4y}{5}. \end{aligned}$$

Similarly we get the lower bound for $-1 - \sqrt{y} \leq u \leq -1$, $-1 \leq u \leq -1 + \sqrt{y}$ and for $1 \leq u \leq 1 + \sqrt{y}$. This inequality proves the Lemma. \square

Lemma 8.11. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $c_0 > 0$ such that for any $\mathbb{J} \subset \mathbb{T}$,*

$$|z + ym_n^{(\mathbb{J})}(z) + ys_y(z) + \frac{y-1}{z}| \geq y \operatorname{Im} m_n^{(\mathbb{J})}(z) + \operatorname{Im} \left\{ \frac{y-1}{z} \right\}, \quad (8.29)$$

moreover, for $z \in \mathbb{G}$,

$$|z + ym_n^{(\mathbb{J})}(z) + ys_y(z) + \frac{y-1}{z}| \geq c_0 \sqrt{\left| \left(z + \frac{y-1}{z} \right)^2 - 4y \right|}. \quad (8.30)$$

Proof. Firstly we note

$$\begin{aligned} |z + y(m_n^{(\mathbb{J})}(z) + s_y(z)) + \frac{y-1}{z}| &\geq \operatorname{Im} \left\{ ys_y(z) + z + \frac{y-1}{z} \right\} \\ &\geq \frac{1}{2} \operatorname{Im} \sqrt{\left(z + \frac{y-1}{z} \right)^2 - 4y}. \end{aligned}$$

Furthermore, it is simple to check that, for $z \in \mathbb{G}$

$$\operatorname{Re} \left\{ \left(z + \frac{y-1}{z} \right)^2 - 4y \right\} \leq 0.$$

This implies that

$$\operatorname{Im} \sqrt{\left(z + \frac{y-1}{z} \right)^2 - 4y} \geq \frac{\sqrt{2}}{2} \sqrt{\left| \left(z + \frac{y-1}{z} \right)^2 - 4y \right|}.$$

Thus Lemma 8.11 is proved. \square

Lemma 8.12. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \dots, n$,*

$$\mathbf{E}\{|\varepsilon_{j2}|^2 | \mathfrak{M}^{(j)}\} \leq \frac{C}{n} (v^{-1} \operatorname{Im} m_n^{(j)}(z) + \frac{1-y}{|z|^2}), \quad (8.31)$$

and

$$\mathbf{E}\{|\varepsilon_{j2}|^4 | \mathfrak{M}^{(j)}\} \leq \frac{C\mu_4^2}{n^2} (v^{-1} \operatorname{Im} m_n^{(j)}(z) + \frac{1-y}{|z|^2})^2. \quad (8.32)$$

Proof. Note that r.v.'s X_{jl} , for $l = 1, \dots, p$ are independent of $\mathfrak{M}^{(j)}$ and that for $l, k = 1, \dots, p$, $R_{l+n, k+n}^{(j)}$ are measurable with respect to $\mathfrak{M}^{(j)}$. This implies that ε_{j2} is a quadratic form with coefficients $R_{l+n, l+n}^{(j)}$ independent of X_{jl} . Thus its variance and fourth moment are easily available.

$$\mathbf{E}\{|\varepsilon_{j2}|^2 | \mathfrak{M}^{(j)}\} = \frac{1}{n^2} \sum_{l,k=1}^p |R_{l+n, k+n}^{(j)}|^2 \leq \frac{1}{n^2} \operatorname{Tr} |\mathbf{R}^{(j)}|^2,$$

Here we use the notation $|\mathbf{A}|^2 = \mathbf{A}\mathbf{A}^*$ for any matrix \mathbf{A} . Applying Lemma 8.4, inequality (8.6), we get equality (8.31).

Furthermore, direct calculations show that

$$\begin{aligned} \mathbf{E}\{|\varepsilon_{j2}|^4|\mathfrak{M}^{(j)}\} &\leq \frac{C}{n^2} \left(\frac{1}{n} \sum_{1 \leq l \neq k \leq p} |R_{l+n,k+n}^{(j)}|^2\right)^2 + \frac{C\mu_4^2}{n^2} \frac{1}{n^2} \sum_{l,k=1}^p |R_{l+n,k+n}^{(j)}|^4 \\ &\leq \frac{C\mu_4^2}{n^2} \left(\frac{1}{n} \sum_{1 \leq l \neq k \leq p} |R_{l+n,k+n}^{(j)}|^2\right)^2 \leq \frac{C\mu_4^2}{n^2} (v^{-1} \operatorname{Im} m_n^{(j)}(z) + \frac{1-y}{|z|^2})^2. \end{aligned}$$

Here again we used Lemma 8.4, inequality (8.6). Thus Lemma 8.12 is proved. \square

Lemma 8.13. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \dots, n$,*

$$\mathbf{E}\{|\varepsilon_{j1}|^2|\mathfrak{M}^{(j)}\} \leq \frac{C\mu_4}{n} \frac{1}{n} \sum_{l=1}^p |R_{l+n,l+n}^{(j)}|^2, \quad (8.33)$$

and

$$\mathbf{E}\{|\varepsilon_{j1}|^4|\mathfrak{M}^{(j)}\} \leq \frac{C\mu_4}{n^2} \frac{1}{n} \sum_{l=1}^p |R_{l+n,l+n}^{(j)}|^4. \quad (8.34)$$

Proof. The first inequality is obvious. To prove the second inequality, we apply Rosenthal's inequality. We obtain

$$\mathbf{E}\{|\varepsilon_{j1}|^4|\mathfrak{M}^{(j)}\} \leq \frac{C\mu_4}{n^2} \left(\frac{1}{n} \sum_{l=1}^p |R_{l+n,l+n}^{(j)}|^2\right)^2 + \frac{C\mu_8}{n^3} \frac{1}{n} \sum_{l=1}^p |R_{l+n,l+n}^{(j)}|^4.$$

Using $|X_{jl}| \leq Cn^{\frac{1}{4}}$ we get $\mu_8 \leq Cn\mu_4$ and the claim. Thus Lemma 8.13 is proved. \square

Corollary 8.14. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$, depending on μ_4 and D only, such that for any $j = 1, \dots, n$, $\nu = 1, 2$, $z \in \mathbb{G}$, and $1 \leq \alpha \leq \frac{1}{2}A_1(n\nu)^{\frac{1}{4}}$,*

$$\mathbf{E} \frac{|\varepsilon_{j\nu}|^2}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}| |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^\alpha} \leq \frac{C}{n\nu} \quad (8.35)$$

and

$$\mathbf{E} \frac{|\varepsilon_{j\nu}|^4}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^2 |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^\alpha} \leq \frac{C}{n^2\nu^2}. \quad (8.36)$$

Proof. For $\nu = 1$, by Lemma 8.11, we have

$$\begin{aligned} & \mathbf{E} \frac{|\varepsilon_{j1}|^2}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{1-y}{z}| |z + ym_n^{(j)}(z) + \frac{1-y}{z}|^\alpha} \\ & \leq \mathbf{E} \frac{y \operatorname{Im} m_n^{(j)} + \frac{C(1-y)v}{|z|}}{nv |z + y(m_n^{(j)}(z) + s(z)) + \frac{1-y}{z}| |z + ym_n^{(j)}(z) + \frac{1-y}{z}|^\alpha}. \end{aligned} \quad (8.37)$$

Note that, for $z \in \mathbb{G}$,

$$y \operatorname{Im} m_n^{(j)} + \frac{C(1-y)v}{|z|} \leq |z + y(m_n^{(j)}(z) + s(z)) + \frac{1-y}{z}|. \quad (8.38)$$

This inequality and inequality (8.37) together imply

$$\mathbf{E} \frac{|\varepsilon_{j1}|^2}{|z + ym_n^{(j)}(z) + \frac{y-1}{z} + s_y(z)| |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^\alpha} \leq \frac{C}{nv} \mathbf{E} \frac{1}{|z + ym_n^{(j)}(z) + \frac{1-y}{z}|^\alpha}.$$

Applying now Corollary 6.15, we get the claim. The proof of the second inequality for $\nu = 1$ is similar. For $\nu = 2$ we apply Lemma 8.12, inequality (8.31) and obtain, using (8.38),

$$\mathbf{E} \frac{|\varepsilon_{j2}|^2}{|z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}| |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^\alpha} \leq \frac{C}{nv}.$$

Similarly, using Lemma 8.12, inequality (8.32), we get

$$\begin{aligned} & \mathbf{E} \frac{|\varepsilon_{j2}|^4}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^2 |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^\alpha} \\ & \leq \frac{C}{n^2 v^2} \mathbf{E} \frac{(\operatorname{Im} m_n^{(j)}(z) + \frac{(1-y)v}{|z|^2})^2}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^2 |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^\alpha} \\ & \leq \frac{C}{n^2 v^2} \mathbf{E} \frac{1}{|z + ym_n^{(j)}(z) + \frac{y-1}{z}|^\alpha}. \end{aligned}$$

Applying Corollary 6.15, we get the claim. \square

Lemma 8.15. *Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$ such that for any $j = 1, \dots, n$,*

$$|\varepsilon_{j3}| \leq \frac{C}{nv} \quad a.s. \quad (8.39)$$

Proof. Firstly we represent

$$\varepsilon_{j3} = \frac{1}{2n} \text{Tr } \mathbf{R} - \frac{1}{2n} \text{Tr } \mathbf{R}^{(j)} + \frac{1-y}{2z}. \quad (8.40)$$

This equality implies

$$|\varepsilon_{j3}| \leq \frac{1}{nv}.$$

□

Lemma 8.16. *Assuming the conditions of Theorem 1.1, we have, for $z \in \mathbb{G}$,*

$$\mathbf{E}|\Lambda_n|^2 \leq \frac{C}{nv|(z + \frac{y-1}{z})^2 - 4|^{\frac{1}{2}}}.$$

Proof. We write

$$\begin{aligned} \mathbf{E}|\Lambda_n|^2 &= \mathbf{E}\Lambda_n \bar{\Lambda}_n = \mathbf{E} \frac{T_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \bar{\Lambda}_n \\ &= \sum_{\nu=1}^3 \mathbf{E} \frac{T_{n\nu}}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}} \bar{\Lambda}_n, \end{aligned}$$

where

$$T_{n\nu} := \frac{1}{n} \sum_{j=1}^n \varepsilon_{j\nu} R_{jj}, \text{ for } \nu = 1, \dots, 3.$$

Applying Cauchy – Schwartz inequality , we get

$$\mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 \leq \sum_{\nu=1}^3 \mathbf{E}^{\frac{1}{2}} \frac{|T_{n\nu}|^2}{|z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}|^2}. \quad (8.41)$$

First we observe that

$$\begin{aligned} T_{n3} &= \frac{1}{2n^2} \sum_{j=1}^n (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)}) R_{jj} + \frac{1}{2n^2 z} \sum_{j=1}^n R_{jj} \\ &= -\frac{1}{2n} \frac{d}{dz} m_n(z) + \frac{1-y}{2nz} m_n(z). \end{aligned}$$

Therefore,

$$|T_{n3}| = \frac{1}{n} |m'_n(z)| + \frac{1-y}{|z|} |m_n(z)| \leq \frac{1}{nv} \text{Im } m_n(z) + \frac{1-y}{n|z|} |m_n(z)|.$$

Hence $|z + y(m_n(z) + s_y(z)) + \frac{1-y}{z}| \geq \operatorname{Im} m_n(z) + \frac{(1-y)v}{|z|^2}$ and Jensen's inequality yields

$$\mathbf{E} \frac{|T_{n3}|^2}{|z + y(m_n(z) + s_y(z)) + \frac{1-y}{z}|^2} \leq \frac{C}{n^2 v^2} (1 + |z|^2) \leq \frac{4C}{n^2 v^2}. \quad (8.42)$$

Furthermore, we observe that,

$$\begin{aligned} \frac{1}{|z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}|} &\leq \frac{1}{|z + y(s_y(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|} \\ &\quad \times \left(1 + \frac{|\varepsilon_{j3}|}{|z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}|}\right). \end{aligned}$$

Therefore, by Lemmas 8.15 and 8.10, for $z \in \mathbb{G}$,

$$\frac{1}{|z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}|} \leq \frac{C}{|z + y(s_y(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|}. \quad (8.43)$$

Applying inequality (8.43), we may write, for $\nu = 1, 2$

$$\begin{aligned} \mathbf{E} \frac{|T_{n\nu}|^2}{|z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}|^2} \\ \leq \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}|^2 |R_{jj}|^2}{|z + y(s_y(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|^2}. \end{aligned}$$

Applying Cauchy – Schwartz inequality and Lemma 8.11, we get

$$\begin{aligned} \mathbf{E} \frac{|T_{n\nu}|^2}{|z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}|^2} \\ \leq \frac{C}{n|(z + \frac{y-1}{z})^2 - 4|^{\frac{1}{2}}} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j\nu}|^4}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^2} \mathbf{E}^{\frac{1}{2}} |R_{jj}|^4. \end{aligned}$$

Using now Corollary 8.14, inequality (8.36) and Corollary 6.15, we get for $\nu = 1, 2, 3$

$$\mathbf{E} \frac{|T_{n\nu}|^2}{|z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}|^2} \leq \frac{C}{nv|(z + \frac{y-1}{z})^2 - 4|^{\frac{1}{2}}}. \quad (8.44)$$

Inequalities (8.41), (8.42) and (8.44) together complete the proof. Thus Lemma 8.16 is proved. \square

Lemma 8.17. *Assuming the conditions of Theorem 1.1, we have, for $z \in \mathbb{G}$,*

$$\mathbf{E}|\Lambda_n|^2 \leq \frac{C}{n^2 v^2}.$$

Proof. We write

$$\begin{aligned} \mathbf{E}|\Lambda_n|^2 &= \mathbf{E}\Lambda_n \bar{\Lambda}_n = \mathbf{E} \frac{T_n}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}} \bar{\Lambda}_n \\ &= \sum_{\nu=1}^4 \mathbf{E} \frac{T_{n\nu}}{z + y(m_n(z) + s(z)) + \frac{y-1}{z}} \bar{\Lambda}_n, \end{aligned}$$

where

$$T_{n\nu} := \frac{1}{n} \sum_{j=1}^n \varepsilon_{j\nu} R_{jj}, \text{ for } \nu = 1, 2, 3.$$

First we observe that by (6.40)

$$|T_{n3}| = \frac{1}{n} |m'_n(z)| \leq \frac{1}{nv} (\operatorname{Im} m_n(z) + \frac{(1-y)v}{|z|^2}).$$

Hence $|z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}| \geq \operatorname{Im} m_n^{(j)}(z) + \frac{(1-y)v}{|z|^2}$ and Jensen's inequality yields

$$|\mathbf{E} \frac{T_{n3}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \bar{\Lambda}_n| \leq \frac{1}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \quad (8.45)$$

Consider now the quantity

$$Y_\nu := \mathbf{E} \frac{T_{n\nu}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \bar{\Lambda}_n,$$

for $\nu = 1, 2$. We represent it as follows

$$Y_\nu = Y_{\nu 1} + Y_{\nu 2},$$

where

$$\begin{aligned} Y_{\nu 1} &= -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\Lambda}_n}{(z + y m_n^{(j)}(z) + \frac{y-1}{z})(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})}, \\ Y_{\nu 2} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} (R_{jj} + \frac{1}{z + y m_n^{(j)}(z) + \frac{y-1}{z}}) \bar{\Lambda}_n}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}}. \end{aligned}$$

By the representation (4.6), which is similar to (6.32) we have

$$Y_{\nu 2} = \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu}(\varepsilon_{j1} + \varepsilon_{j2})\bar{\Lambda}_n}{(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})(z + ym_n^{(j)}(z) + \frac{y-1}{z})}.$$

Using inequality (8.43), we may write, for $z \in \mathbb{G}$

$$|Y_{\nu 2}| \leq \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\varepsilon_{j1} + \varepsilon_{j2}| |\bar{\Lambda}_n|}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}| |z + ym_n^{(j)}(z) + \frac{y-1}{z}|}.$$

Applying the Cauchy – Schwartz inequality and the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we get

$$\begin{aligned} |Y_{\nu 2}| &\leq \frac{C}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j\nu}|^2 |\varepsilon_{j1} + \varepsilon_{j2}|^2}{|z + ym_n^{(j)}(z) + \frac{y-1}{z} + s(z)|^2 |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^2} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 \\ &\leq \frac{C}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j1} + \varepsilon_{j2}|^4}{|z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}|^2 |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^2} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \end{aligned} \quad (8.46)$$

Using Corollary 8.14 with $\alpha = 2$, we arrive at

$$|Y_{\nu 2}| \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2. \quad (8.47)$$

In order to estimate $Y_{\nu 1}$ we introduce now the quantity

$$\Lambda_n^{(j,1)} = \frac{1}{n} \text{Tr } \mathbf{R}^{(j)} - s_y(z) + \frac{s_y(z)}{2n} + \frac{1}{2nz}.$$

Note that

$$\begin{aligned} \Lambda_n - \Lambda_n^{(j,1)} &= \frac{1}{n} \left(\frac{1}{2} (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)}) + \frac{1}{z} \right) - \frac{s(z)}{2n} - \frac{1}{2nz} \\ &= \frac{R_{jj} - s_y(z)}{2n} + \frac{1}{np} \sum_{l,k=1}^p X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{k+n,l+n} = \delta_{nj}. \end{aligned} \quad (8.48)$$

We represent $Y_{\nu 1}$ in the form

$$Y_{\nu 1} = Z_{\nu 1} + Z_{\nu 2} + Z_{\nu 3},$$

where

$$\begin{aligned} Z_{\nu 1} &= -\frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\Lambda}_n^{(j1)}}{(z + ym_n^{(j)}(z) + \frac{y-1}{z})(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})}, \\ Z_{\nu 2} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\delta}_{nj}}{(z + ym_n^{(j)}(z) + \frac{y-1}{z})(z + y(m_n(z) + s(z)) + \frac{y-1}{z})}, \\ Z_{\nu 3} &= \frac{1}{n} \sum_{j=1}^n \mathbf{E} \frac{\varepsilon_{j\nu} \bar{\Lambda}_n^{(j1)}}{(z + ym_n^{(j)}(z) + \frac{y-1}{z})(z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z})} \\ &\quad \times \frac{\varepsilon_{j3}}{(z + y(m_n(z) + s_y(z)) + \frac{y-1}{z})}. \end{aligned}$$

First, note that by conditional independence

$$Z_{\nu 1} = 0. \quad (8.49)$$

Furthermore, applying Hölder's inequality, we get

$$\begin{aligned} |Z_{\nu 3}| &\leq \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{4}} \frac{|\varepsilon_{j\nu}|^4}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^4 |z + ym_n^{(j)}(z) + \frac{y-1}{z}|^4} \\ &\quad \times \mathbf{E}^{\frac{1}{4}} \frac{|\varepsilon_{j3}|^4}{|z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}|^4} \mathbf{E}^{\frac{1}{2}} |\Lambda_n^{(j,1)}|^2. \end{aligned}$$

Using Corollary 8.14 with $\alpha = 4$ and Lemmas 8.22 and 8.11, we obtain

$$|Z_{\nu 3}| \leq \frac{C}{(nv)^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} |\Lambda_n^{(j,1)}|^2.$$

Applying now Corollary (6.15), we get

$$|Z_{\nu 3}| \leq \frac{C}{(nv)^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{(nv)^{\frac{3}{2}} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} |\delta_{nj}|^2.$$

For $z \in \mathbb{G}$ we may rewrite this bound using Lemma 8.10

$$|Z_{\nu 3}| \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{nv} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} |\delta_{nj}|^2.$$

By definition of δ_{nj} , see (8.48), we have

$$\mathbf{E} |\delta_{nj}|^2 \leq C \left(\frac{1}{n^2} \mathbf{E} |R_{jj} - s(z)|^2 + \mathbf{E}^{\frac{1}{2}} \left| \frac{1}{n^2} \sum_{l,k \in \mathbb{T}_j} X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{k+n, l+n} \right|^4 \mathbf{E}^{\frac{1}{2}} |R_{jj}|^4 \right).$$

By representation (6.32), we have

$$\mathbf{E}|R_{jj} - s(z)|^2 \leq \mathbf{E}|\Lambda_n|^2 + \mathbf{E}^{\frac{1}{2}}|\varepsilon_j|^4 \mathbf{E}^{\frac{1}{2}}|R_{jj}|^4. \quad (8.50)$$

Note that by Lemmas 8.12, 8.32, 8.33, we have

$$\mathbf{E}|\varepsilon_j|^4 \leq \frac{C}{n^2 v^2}. \quad (8.51)$$

By Corollaries 6.15 and 8.51, we get

$$\mathbf{E}|R_{jj} - s(z)|^2 \leq \mathbf{E}|\Lambda_n|^2 + \frac{C}{nv}. \quad (8.52)$$

By Lemmas 8.20, inequality (8.63) and Corollary 6.15, we have

$$\mathbf{E} \left| \frac{1}{n^2} \sum_{l,k \in \mathbb{T}_j} X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{k+n,l+n} \right|^4 = \frac{1}{n^4} \mathbf{E} |\eta_{j3}|^4 \leq \frac{C}{n^6 v^6}. \quad (8.53)$$

Inequalities (8.52) and (8.53) together imply

$$\mathbf{E}|\delta_{nj}|^2 \leq \frac{1}{n^2} \mathbf{E}|\Lambda_n|^2 + \frac{C}{n^3 v} + \frac{C}{n^6 v^6}. \quad (8.54)$$

Therefore, for $z \in \mathbb{G}$,

$$|Z_{\nu 3}| \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 + \frac{C}{n^4 v^4} + \frac{1}{n^2 v} \mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}}|\Lambda_n|^2 + \frac{C}{n^2 v^2}. \quad (8.55)$$

To bound $Z_{\nu 2}$ we first apply inequality (8.43) and obtain

$$|Z_{\nu 2}| \leq \frac{C}{n} \sum_{j=1}^n \mathbf{E} \frac{|\varepsilon_{j\nu}| |\delta_{nj}|}{|z + y m_n^{(j)}(z) + \frac{y-1}{z}| |z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|}.$$

Applying now Hölder's inequality, we get

$$|Z_{\nu 2}| \leq \frac{C}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} \frac{|\varepsilon_{j\nu}|^2}{|z + y m_n^{(j)}(z) + \frac{y-1}{z}|^2 |z + y m_n^{(j)}(z) + \frac{y-1}{z} + s(z)|^2} \mathbf{E}^{\frac{1}{2}} |\delta_{nj}|^2.$$

Therefore, by Corollary 8.14

$$|Z_{\nu 2}| \leq \frac{C}{\sqrt{nv} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}} \frac{1}{n} \sum_{j=1}^n \mathbf{E}^{\frac{1}{2}} |\delta_{nj}|^2.$$

The last inequality together with Lemma 8.10 and inequality (8.54), imply

$$\begin{aligned} |Z_{\nu 2}| &\leq \frac{C}{n\sqrt{nv} |(z + \frac{y-1}{z})^2 - 4y|^{\frac{1}{4}}} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v} + \frac{C}{n^{\frac{7}{2}} v^{\frac{7}{2}}} \\ &\leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \end{aligned} \quad (8.56)$$

Combining now inequalities (8.45), (8.49), (8.55), (8.56), we get

$$\mathbf{E} |\Lambda_n|^2 \leq \frac{C}{nv} \mathbf{E}^{\frac{1}{2}} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \quad (8.57)$$

Solving this inequality with respect to $\mathbf{E} |\Lambda_n|^2$ completes the proof of Lemma 8.17. Thus Lemma 8.17 is proved. \square

Lemma 8.18. *There exists a positive constant C such that*

$$|\delta_{n3}| \leq \frac{1}{nv} \text{Im } m_n(z). \quad (8.58)$$

Proof. It is easy to check that

$$\sum_{k=1}^p (R_{k+n, k+n} - R_{k+n, k+n}^{(j)}) = \frac{1}{2} (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)}) + \frac{1}{2z}.$$

By formula (5.4) in [9], we have

$$(\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)}) R_{jj} = (1 + \frac{1}{p} \sum_{l,k=1}^n X_{jl} X_{jk} (R^{(j)})_{l+n, k+n}^2) R_{jj}^2 = -\frac{d}{dz} R_{jj}.$$

From here it follows that

$$\frac{1}{n^2} \sum_{j=1}^n (\text{Tr } \mathbf{R} - \text{Tr } \mathbf{R}^{(j)}) R_{jj} = -\frac{1}{n} \frac{d}{dz} m_n(z).$$

Note that

$$m_n(z) = \frac{z}{n} \sum_{k=1}^n \frac{1}{s_k^2 - z^2}$$

and

$$\frac{d}{dz} m_n(z) = \frac{m_n(z)}{z} - \frac{2z^2}{n} \sum_{k=1}^n \frac{1}{(s_k^2 - z^2)^2}.$$

This implies that

$$\delta_{n3} = \frac{1}{2n}(-m'_n(z) + \frac{m_n(z)}{z}) = \frac{z^2}{n^2} \sum_{k=1}^n \frac{1}{(s_k^2 - z^2)^2}. \quad (8.59)$$

Finally, we note that

$$\operatorname{Im} m_n(z) = \frac{1}{n} \sum_{k=1}^n \frac{v(s_k^2 + |z|^2)}{|s_k^2 - z^2|^2}. \quad (8.60)$$

The last relation implies

$$\left| \frac{z^2}{n^2} \sum_{k=1}^n \frac{1}{(s_k^2 - z^2)^2} \right| \leq \frac{1}{nv} \operatorname{Im} m_n(z). \quad (8.61)$$

The inequality (8.61) concludes the proof. Thus Lemma 8.18 is proved. \square

We introduce the following quantity

$$\begin{aligned} \beta_{j1} &= \frac{1}{p} \sum_{l=1}^p [(\mathbf{R}^{(j)})^2]_{l+n, l+n} - \frac{1}{p} \sum_{l=1}^p [(\mathbf{R})^2]_{l+n, l+n}, \\ \beta_{j2} &= \frac{1}{p} \sum_{1 \leq l \neq k \leq p} X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{l+n, k+n}, \\ \beta_{j3} &= \frac{1}{n} \sum_{l=1}^p (X_{jl}^2 - 1) [(\mathbf{R}^{(j)})^2]_{l+n, l+n}. \end{aligned}$$

Lemma 8.19. *Assuming the conditions of Theorem 1.1, we have, for $\nu = 2, 3$,*

$$\mathbf{E}\{|\beta_{j\nu}|^2 | \mathfrak{M}^{(j)}\} \leq \frac{C}{nv^3} \operatorname{Im} m_n^{(j)}(z). \quad (8.62)$$

Proof. We recall that by C we denote a generic constant depending on μ_4 and D only. By definition of $\beta_{j\nu}$ for $\nu = 1, 2$, conditioning on $\mathfrak{M}^{(j)}$, we get

$$\begin{aligned} \mathbf{E}\{|\beta_{j2}|^2 | \mathfrak{M}^{(j)}\} &\leq \frac{C}{n^2} \sum_{1 \leq l \neq k \leq p} |[(\mathbf{R}^{(j)})^2]_{k+n, l+n}|^2 \leq \frac{C}{n^2} \sum_{1 \leq l, k \leq p} |[(\mathbf{R}^{(j)})^2]_{k+n, l+n}|^2, \\ \mathbf{E}\{|\beta_{j3}|^2 | \mathfrak{M}^{(j)}\} &\leq \frac{C}{n^2} \sum_{l=1}^p |[(\mathbf{R}^{(j)})^2]_{l+n, l+n}|^2 \leq \frac{C}{n^2} \sum_{l, k=1}^p |[(\mathbf{R}^{(j)})^2]_{k+n, l+n}|^2. \end{aligned}$$

Applying Lemma 8.4, we get the claim. Thus Lemma 8.19 is proved. \square

Lemma 8.20. *Assuming the conditions of Theorem 1.1, we have,*

$$\mathbf{E}\{|\beta_{j\nu}|^8 \mid \mathfrak{M}^{(j)}\} \leq \frac{C}{n^4 v^{12}}. \quad (8.63)$$

Proof. Let C denote a generic constant depending on μ_4 and D only. First we consider the case $q = 8$. By definition of $\beta_{j\nu}$ for $\nu = 2, 3$, conditioning on $\mathfrak{M}^{(j)}$, direct calculations shows

$$\begin{aligned} \mathbf{E}\{|\beta_{j2}|^8 \mid \mathfrak{M}^{(j)}\} &\leq \frac{C}{n^8} \left(\left(\sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^2 \right)^4 + \mu_4^4 \left(\sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^4 \right)^2 \right. \\ &\quad + \mu_6^2 \left(\sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^6 \right) \mu_8^2 \left(\sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^2 \right) \\ &\quad \left. + \mu_8^2 \sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^8 \right). \end{aligned}$$

Using that $\mu_6 \leq C\sqrt{n}\mu_4$ and $\mu_8 \leq Cn\mu_4$, we get

$$\begin{aligned} \mathbf{E}\{|\beta_{j2}|^8 \mid \mathfrak{M}^{(j)}\} &\leq \frac{C}{n^8} \left(\left(\sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^2 \right)^4 \right. \\ &\quad + n \left(\sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^6 \right) \left(\sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^2 \right) \\ &\quad \left. + n^2 \sum_{1 \leq l \neq k \leq p} |[(R^{(j)})^2]_{k+n, l+n}|^8 \right). \end{aligned}$$

Applying Lemma 8.4, inequalities (8.16) and (8.17) and Corollary 6.15, we get the claim. Thus Lemma 8.20 is proved. \square

Lemma 8.21. *Assuming the conditions of Theorem 1.1, we have, for $j = 1, \dots, n$,*

$$|\beta_{j1}| \leq \frac{C}{nv^2}.$$

Proof. Let $\mathcal{F}_n^{(j)}(x)$ denote empirical spectral distribution function of the matrix $\mathbf{W}^{(j)}$. According to *the interlacing eigenvalues Theorem* (see [21], Theorem 4.38) we have

$$\sup_x |\mathcal{F}_n(x) - \mathcal{F}_n^{(j)}(x)| \leq \frac{C}{n}. \quad (8.64)$$

Furthermore, we represent

$$\beta_{j1} = \int_{-\infty}^{\infty} \frac{1}{(x-z)^2} d(\mathcal{F}_n(x) - \mathcal{F}_n^{(j)}(x)) + \frac{1}{n|z|^2}.$$

Integrating by parts, we get the claim.

Thus Lemma 8.21 is proved. \square

Lemma 8.22. *Assuming the conditions of Theorem 1.1, we have*

$$\mathbf{E} \frac{|\varepsilon_{j3}|^4}{|z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}|^4} \leq \frac{C(y)}{n^4 v^4}. \quad (8.65)$$

Proof. Using the representations (6.40) we have

$$\begin{aligned} \varepsilon_{j3} &= \frac{y}{2n} \left(1 + \frac{1}{p} \sum_{l,k=1}^p X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{l+n,k+n} \right) R_{jj} + \frac{y}{2nz} \\ &= \frac{y}{2n} \left((1 + \beta_{j1} + \beta_{j2} + \beta_{j3}) R_{jj} + \frac{1}{2z} \right). \end{aligned} \quad (8.66)$$

Applying the Cauchy–Schwartz inequality, we get

$$\begin{aligned} &\mathbf{E} \left| \frac{\varepsilon_{j3}}{z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}} \right|^4 \\ &\leq \frac{C}{n^4} \left(\left(1 + \mathbf{E}^{\frac{1}{2}} \left(\frac{|\frac{1}{n} \sum_{l,k=1}^p X_{jl} X_{jk} [(\mathbf{R}^{(j)})^2]_{l+n,k+n}|}{|z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}|} \right)^8 \right) \mathbf{E}^{\frac{1}{2}} |R_{jj}|^8 + \frac{1}{|z|^4} \right). \end{aligned}$$

Using Corollary 6.15, we we may write

$$\begin{aligned} &\mathbf{E} \left| \frac{\varepsilon_{j3}}{z + y(s_y(z) + m_n(z)) + \frac{y-1}{z}} \right|^4 \\ &\leq \frac{C}{n^4} \left(1 + \frac{1}{|z|^4} + \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j1}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 \right. \\ &\quad \left. + \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j2}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 + \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j3}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 \right). \end{aligned}$$

Using Lemma 8.21, we get by definition of η_{j1} , Lemma 8.4, and inequality (8.6) that for $z \in \mathbb{G}$,

$$\begin{aligned} \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j1}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 &\leq C \mathbf{E}^{\frac{1}{2}} \frac{C}{n^8 v^{16} |z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^8} \\ &\leq C v^{-4}. \end{aligned} \quad (8.67)$$

Furthermore, applying inequality (8.43), we obtain

$$\mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j2}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 \leq \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j2}}{z + y(m_n^{(j)}(z) + s(z)) + \frac{y-1}{z}} \right|^8.$$

Conditioning with respect to $\mathfrak{M}^{(j)}$ and applying Lemma 8.19, we obtain

$$\begin{aligned} & \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j2}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 \\ & \leq \mathbf{E}^{\frac{1}{2}} \frac{C}{|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}|^8} \left(\frac{1}{n^4 v^{12}} (\operatorname{Im} m_n^{(j)}(z))^4 \right. \\ & \quad \left. + \frac{\mu_4^4}{n^4 v^8} \frac{1}{n} \sum_{l=1}^p (\operatorname{Im} R_{l+n, l+n}^{(j)})^8 \right). \end{aligned} \quad (8.68)$$

Using Lemma 8.11, inequality (8.30), together with Corollary 6.15 we get

$$\begin{aligned} \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j2}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 & \leq \frac{C}{n^2 v^6 |(z + \frac{y-1}{z})^2 - 4y|} \\ & \quad + \frac{C \mu_4^2}{n^2 v^4 |(z + \frac{y-1}{z})^2 - 4y|^2}. \end{aligned}$$

Applying inequality (8.43) and conditioning with respect to $\mathfrak{M}^{(j)}$ and applying Lemma 8.19, we get

$$\begin{aligned} & \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j3}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 \\ & \leq \mathbf{E}^{\frac{1}{2}} \frac{1}{|z + y(s_y(z) + m_n^{(j)}(z)) + \frac{y-1}{z}|^8} \left(\frac{C}{n^4 v^{12}} (\operatorname{Im} m_n^{(j)}(z))^4 \right. \\ & \quad + \frac{C \mu_4^2}{n^4 v^8} \left(\frac{1}{n} \sum_{l \in \mathbb{T}_j} (\operatorname{Im} R_{ll}^{(j)})^4 \right)^2 \\ & \quad + \frac{C \mu_4^2}{n^5 v^{12}} \left(\frac{1}{n} \sum_{l \in \mathbb{T}_j} (\operatorname{Im} R_{ll}^{(j)})^3 \right) (\operatorname{Im} m_n^{(j)}(z)) \\ & \quad + \frac{C \mu_4^4}{n^6 v^{12}} \left(\frac{1}{n} \sum_{l \in \mathbb{T}_j} (\operatorname{Im} R_{ll}^{(j)})^2 \right)^2 \\ & \quad \left. + \frac{C \mu_4^2}{n^6 v^{12}} \left(\frac{1}{n} \sum_{l \in \mathbb{T}_j} (\operatorname{Im} R_{ll}^{(j)})^2 \right) (\operatorname{Im} m_n^{(j)}(z))^2 \right). \end{aligned} \quad (8.69)$$

Using that $|z + y(m_n^{(j)}(z) + s_y(z)) + \frac{y-1}{z}| \geq \operatorname{Im} m_n^{(j)}(z) + \frac{(1-y)v}{|z|^2}$ together with Lemma 8.4, we arrive at

$$\begin{aligned} \mathbf{E}^{\frac{1}{2}} \left| \frac{\beta_{j3}}{z + y(m_n(z) + s_y(z)) + \frac{y-1}{z}} \right|^8 &\leq \frac{C}{n^2 v^6 |(z + \frac{y-1}{z})^2 - 4y|} \\ &+ \frac{C}{n^2 v^4 |(z + \frac{y-1}{z})^2 - 4y|^2} + \frac{C}{n^{\frac{5}{2}} v^6 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{7}{4}}} \\ &+ \frac{C}{n^3 v^6 |(z + \frac{y-1}{z})^2 - 4y|^2} + \frac{C}{n^3 v^6 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{3}{2}}}. \end{aligned}$$

Summarizing we may write now, for $z \in \mathbb{G}$,

$$\begin{aligned} \mathbf{E} \frac{|\varepsilon_{j3}|^4}{|z + s(z) + m_n(z)|^4} &\leq \frac{C}{n^4 v^4} + \frac{C}{n^6 v^6 |(z + \frac{y-1}{z})^2 - 4y|} \\ &+ \frac{C}{n^6 v^4 |(z + \frac{y-1}{z})^2 - 4y|^2} + \frac{C}{n^{\frac{13}{2}} v^6 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{7}{4}}} \\ &+ \frac{C}{n^7 v^6 |(z + \frac{y-1}{z})^2 - 4y|^2} + \frac{C}{n^7 v^6 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{3}{2}}}. \end{aligned}$$

For $z \in \mathbb{G}$, see (8.25) and Lemma 8.10, this inequality may be simplified by means of the following bounds (with $v_0 = A_0 n^{-1}$)

$$\begin{aligned} n^{\frac{5}{2}} v^2 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{7}{4}} &\geq n^{\frac{5}{2}} v_0^2 \gamma^{-1+\frac{7}{4}} \geq C \sqrt{n} \gamma^{\frac{3}{4}} \geq C, \\ n^3 v^2 |(z + \frac{y-1}{z})^2 - 4y| &\geq C, \quad n^3 v^2 |(z + \frac{y-1}{z})^2 - 4y|^{\frac{3}{2}} \geq C, \\ n^2 |(z + \frac{y-1}{z})^2 - 4y|^2 &\geq C. \end{aligned} \quad (8.70)$$

Using these relation, we obtain

$$\mathbf{E} \frac{|\varepsilon_{j3}|^4}{|z + y(s(z) + m_n(z)) + \frac{y-1}{z}|^4} \leq \frac{C}{n^4 v^4}.$$

Thus Lemma 8.22 is proved. \square

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